

# Advances in Inequalities of the Schwarz, Grüss and Bessel Type in Inner Product Spaces

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### **Abstract**

The main aim of this monograph is to survey some recent results obtained by the author related to reverses of the Schwarz, triangle and Bessel inequalities. Some Grüss' type inequalities for orthonormal families of vectors in real or complex inner product spaces are presented as well. Generalizations of the Boas-Bellman, Bombieri, Selberg, Heilbronn and Pečarić inequalities for finite sequences of vectors that are not necessarily orthogonal are also provided. Two extensions of the celebrated Ostrowski's inequalities for sequences of real numbers and the generalization of Wagner's inequality in inner product spaces are pointed out. Finally, some Grüss type inequalities for  $n$ -tuples of vectors in inner product spaces and their natural applications for the approximation of the discrete Fourier and Mellin transforms are given as well.

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# Preface

The theory of Hilbert spaces plays a central role in contemporary mathematics with numerous applications for Linear Operators, Partial Differential Equations, in Nonlinear Analysis, Approximation Theory, Optimization Theory, Numerical Analysis, Probability Theory, Statistics and other fields.

The Schwarz, triangle, Bessel, Gram and most recently, Grüss type inequalities have been frequently used as powerful tools in obtaining bounds or estimating the errors for various approximation formulae occurring in the domains mentioned above. Therefore, any new advancement related to these fundamental facts will have a flow of important consequences in the mathematical fields where these inequalities have been used before.

The main aim of this monograph is to survey some recent results obtained by the author related to reverses of the Schwarz, triangle and Bessel inequalities. Some Grüss type inequalities for orthonormal families of vectors in real or complex inner product spaces are presented as well. Generalizations of the Boas-Bellman, Bombieri, Selberg, Heilbronn and Pečarić inequalities for finite sequences of vectors that are not necessarily orthogonal are also provided. Two extensions of the celebrated Ostrowski inequalities for sequences of real numbers and the generalization of Wagner's inequality in inner product spaces are pointed out. Finally, some Grüss type inequalities for  $n$ -tuples of vectors in inner product spaces and their natural applications for the approximation of the discrete Fourier and Mellin transforms are given as well.

The monograph may be used by researchers in different branches of Mathematical and Functional Analysis where the theory of Hilbert spaces is of relevance. Since it is self-contained and all the results are completely proved, the work may be also used by graduate students interested in Theory of Inequalities and its Applications.

Every section is followed by the necessary references and in this way each section may be read independently of the others.

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# Chapter 1

## Reverses for the Schwarz Inequality

### 1.1 Introduction

Let  $H$  be a linear space over the real or complex number field  $\mathbb{K}$ . The functional  $\langle \cdot, \cdot \rangle : H \times H \rightarrow \mathbb{K}$  is called an *inner product* on  $H$  if it satisfies the conditions

- (i)  $\langle x, x \rangle \geq 0$  for any  $x \in H$  and  $\langle x, x \rangle = 0$  iff  $x = 0$ ;
- (ii)  $\langle \alpha x + \beta y, z \rangle = \alpha \langle x, z \rangle + \beta \langle y, z \rangle$  for any  $\alpha, \beta \in \mathbb{K}$  and  $x, y, z \in H$ ;
- (iii)  $\langle y, x \rangle = \overline{\langle x, y \rangle}$  for any  $x, y \in H$ .

A first fundamental consequence of the properties (i)-(iii) above, is the *Schwarz inequality*:

$$|\langle x, y \rangle|^2 \leq \langle x, x \rangle \langle y, y \rangle \quad (1.1)$$

for any  $x, y \in H$ . The equality holds in (1.1) if and only if the vectors  $x$  and  $y$  are *linearly dependent*, i.e., there exists a nonzero constant  $\alpha \in \mathbb{K}$  so that  $x = \alpha y$ .

If we denote  $\|x\| := \sqrt{\langle x, x \rangle}$ ,  $x \in H$ , then one may state the following properties

- (n)  $\|x\| \geq 0$  for any  $x \in H$  and  $\|x\| = 0$  iff  $x = 0$ ;
- (nn)  $\|\alpha x\| = |\alpha| \|x\|$  for any  $\alpha \in \mathbb{K}$  and  $x \in H$ ;
- (nnn)  $\|x + y\| \leq \|x\| + \|y\|$  for any  $x, y \in H$  (the triangle inequality);

i.e.,  $\|\cdot\|$  is a *norm* on  $H$ .

In this chapter we present some recent reverse inequalities for the Schwarz and the triangle inequalities. More precisely, we point out upper bounds for the nonnegative quantities

$$\|x\| \|y\| - |\langle x, y \rangle|, \quad \|x\|^2 \|y\|^2 - |\langle x, y \rangle|^2$$

and

$$\|x\| + \|y\| - \|x + y\|$$

under various assumptions for the vectors  $x, y \in H$ .

If the vectors  $x, y \in H$  are not *orthogonal*, i.e.,  $\langle x, y \rangle \neq 0$ , then some upper bounds for the supra-unitary quantities

$$\frac{\|x\| \|y\|}{|\langle x, y \rangle|}, \quad \frac{\|x\|^2 \|y\|^2}{|\langle x, y \rangle|^2}$$

are provided as well.

## 1.2 An Additive Reverse of the Schwarz Inequality

### 1.2.1 Introduction

Let  $\bar{\mathbf{a}} = (a_1, \dots, a_n)$  and  $\bar{\mathbf{b}} = (b_1, \dots, b_n)$  be two positive  $n$ -tuples with

$$0 < m_1 \leq a_i \leq M_1 < \infty \text{ and } 0 < m_2 \leq b_i \leq M_2 < \infty; \quad (1.2)$$

for each  $i \in \{1, \dots, n\}$ , and some constants  $m_1, m_2, M_1, M_2$ .

The following reverses of the Cauchy-Bunyakovsky-Schwarz inequality for positive sequences of real numbers are well known:

1. *Pólya-Szegő's inequality* [9]

$$\frac{\sum_{k=1}^n a_k^2 \sum_{k=1}^n b_k^2}{\left(\sum_{k=1}^n a_k b_k\right)^2} \leq \frac{1}{4} \left( \sqrt{\frac{M_1 M_2}{m_1 m_2}} + \sqrt{\frac{m_1 m_2}{M_1 M_2}} \right)^2.$$

2. *Shisha-Mond's inequality* [10]

$$\frac{\sum_{k=1}^n a_k^2}{\sum_{k=1}^n a_k b_k} - \frac{\sum_{k=1}^n a_k b_k}{\sum_{k=1}^n b_k^2} \leq \left[ \left( \frac{M_1}{m_2} \right)^{\frac{1}{2}} - \left( \frac{m_1}{M_2} \right)^{\frac{1}{2}} \right]^2.$$

3. *Ozeki's inequality* [8]

$$\sum_{k=1}^n a_k^2 \sum_{k=1}^n b_k^2 - \left( \sum_{k=1}^n a_k b_k \right)^2 \leq \frac{n^2}{4} (M_1 M_2 - m_1 m_2)^2.$$

4. *Diaz-Metcalf's inequality* [1]

$$\sum_{k=1}^n b_k^2 + \frac{m_2 M_2}{m_1 M_1} \sum_{k=1}^n a_k^2 \leq \left( \frac{M_2}{m_1} + \frac{m_2}{M_1} \right) \sum_{k=1}^n a_k b_k.$$

If  $\overline{\mathbf{w}} = (w_1, \dots, w_n)$  is a positive sequence, then the following weighted inequalities also hold:

1. *Cassel's inequality* [11]. If the positive real sequences  $\overline{\mathbf{a}} = (a_1, \dots, a_n)$  and  $\overline{\mathbf{b}} = (b_1, \dots, b_n)$  satisfy the condition

$$0 < m \leq \frac{a_k}{b_k} \leq M < \infty \text{ for each } k \in \{1, \dots, n\}, \quad (1.3)$$

then

$$\frac{(\sum_{k=1}^n w_k a_k^2)(\sum_{k=1}^n w_k b_k^2)}{(\sum_{k=1}^n w_k a_k b_k)^2} \leq \frac{(M+m)^2}{4mM}.$$

2. *Greub-Reinboldt's inequality* [5]. We have

$$\left(\sum_{k=1}^n w_k a_k^2\right) \left(\sum_{k=1}^n w_k b_k^2\right) \leq \frac{(M_1 M_2 + m_1 m_2)^2}{4m_1 m_2 M_1 M_2} \left(\sum_{k=1}^n w_k a_k b_k\right)^2,$$

provided  $\overline{\mathbf{a}} = (a_1, \dots, a_n)$  and  $\overline{\mathbf{b}} = (b_1, \dots, b_n)$  satisfy the condition (1.2).

3. *Generalised Diaz-Metcalf's inequality* [1], see also [7, p. 123]. If  $u, v \in [0, 1]$  and  $v \leq u$ ,  $u + v = 1$  and (1.3) holds, then one has the inequality

$$u \sum_{k=1}^n w_k b_k^2 + v M m \sum_{k=1}^n w_k a_k^2 \leq (v m + u M) \sum_{k=1}^n w_k a_k b_k.$$

4. *Klamkin-McLenaghan's inequality* [6]. If  $\overline{\mathbf{a}}, \overline{\mathbf{b}}$  satisfy (1.3), then

$$\begin{aligned} & \left(\sum_{i=1}^n w_i a_i^2\right) \left(\sum_{i=1}^n w_i b_i^2\right) - \left(\sum_{i=1}^n w_i a_i b_i\right)^2 \\ & \leq \left(M^{\frac{1}{2}} - m^{\frac{1}{2}}\right)^2 \sum_{i=1}^n w_i a_i b_i \sum_{i=1}^n w_i a_i^2. \end{aligned} \quad (1.4)$$

For other recent results providing discrete reverse inequalities, see the recent monograph online [3].

In this section, by following [4], we point out a new reverse of Schwarz's inequality in real or complex inner product spaces. Particular cases for isotonic linear functionals, integrals and sequences are also given.

### 1.2.2 An Additive Reverse Inequality

The following reverse of Schwarz's inequality in inner product spaces holds [4].

**Theorem 1** Let  $A, a \in \mathbb{K}$  ( $\mathbb{K} = \mathbb{C}, \mathbb{R}$ ) and  $x, y \in H$ . If

$$\operatorname{Re} \langle Ay - x, x - ay \rangle \geq 0, \quad (1.5)$$

or equivalently,

$$\left\| x - \frac{a + A}{2} \cdot y \right\| \leq \frac{1}{2} |A - a| \|y\|, \quad (1.6)$$

holds, then one has the inequality

$$0 \leq \|x\|^2 \|y\|^2 - |\langle x, y \rangle|^2 \leq \frac{1}{4} |A - a|^2 \|y\|^4. \quad (1.7)$$

The constant  $\frac{1}{4}$  is sharp in (1.7).

**Proof.** The equivalence between (1.5) and (1.6) can be easily proved, see for example [2].

Let us define

$$I_1 := \operatorname{Re} \left[ \left( A \|y\|^2 - |\langle x, y \rangle| \right) \left( \overline{\langle x, y \rangle} - \bar{a} \|y\|^2 \right) \right]$$

and

$$I_2 := \|y\|^2 \operatorname{Re} \langle Ay - x, x - ay \rangle.$$

Then

$$I_1 = \|y\|^2 \operatorname{Re} \left[ A \overline{\langle x, y \rangle} + \bar{a} \langle x, y \rangle \right] - |\langle x, y \rangle|^2 - \|y\|^4 \operatorname{Re} (A \bar{a})$$

and

$$I_2 = \|y\|^2 \operatorname{Re} \left[ A \overline{\langle x, y \rangle} + \bar{a} \langle x, y \rangle \right] - \|x\|^2 \|y\|^2 - \|y\|^4 \operatorname{Re} (A \bar{a}),$$

which gives

$$I_1 - I_2 = \|x\|^2 \|y\|^2 - |\langle x, y \rangle|^2$$

for any  $x, y \in H$  and  $a, A \in \mathbb{K}$ . This is an interesting identity in itself as well.

If (1.5) holds, then  $I_2 \geq 0$  and thus

$$\|x\|^2 \|y\|^2 - |\langle x, y \rangle|^2 \leq \operatorname{Re} \left[ \left( A \|y\|^2 - |\langle x, y \rangle| \right) \left( \overline{\langle x, y \rangle} - \bar{a} \|y\|^2 \right) \right]. \quad (1.8)$$

Further, if we use the elementary inequality for  $u, v \in \mathbb{K}$  ( $\mathbb{K} = \mathbb{C}, \mathbb{R}$ )

$$\operatorname{Re} [u \bar{v}] \leq \frac{1}{4} |u + v|^2,$$

then we have, for

$$u := A \|y\|^2 - \langle x, y \rangle, \quad v := \langle x, y \rangle - a \|y\|^2,$$

that

$$\operatorname{Re} \left[ \left( A \|y\|^2 - |\langle x, y \rangle| \right) \left( \overline{\langle x, y \rangle} - \bar{a} \|y\|^2 \right) \right] \leq \frac{1}{4} |A - a|^2 \|y\|^4. \quad (1.9)$$

Making use of the inequalities (1.8) and (1.9), we deduce (1.7).

Now, assume that (1.7) holds with a constant  $C > 0$ , i.e.,

$$\|x\|^2 \|y\|^2 - |\langle x, y \rangle|^2 \leq C |A - a|^2 \|y\|^4, \quad (1.10)$$

where  $x, y, a, A$  satisfy (1.5).

Consider  $y \in H$ ,  $\|y\| = 1$ ,  $a \neq A$  and  $m \in H$ ,  $\|m\| = 1$  with  $m \perp y$ . Define

$$x := \frac{A+a}{2}y + \frac{A-a}{2}m.$$

Then

$$\langle Ay - x, x - ay \rangle = \left| \frac{A-a}{2} \right|^2 \langle y - m, y + m \rangle = 0,$$

and thus the condition (1.5) is fulfilled. From (1.10) we deduce

$$\left\| \frac{A+a}{2}y + \frac{A-a}{2}m \right\|^2 - \left| \left\langle \frac{A+a}{2}y + \frac{A-a}{2}m, y \right\rangle \right|^2 \leq C |A-a|^2, \quad (1.11)$$

and since

$$\left\| \frac{A+a}{2}y + \frac{A-a}{2}m \right\|^2 = \left| \frac{A+a}{2} \right|^2 + \left| \frac{A-a}{2} \right|^2$$

and

$$\left| \left\langle \frac{A+a}{2}y + \frac{A-a}{2}m, y \right\rangle \right|^2 = \left| \frac{A+a}{2} \right|^2$$

then, by (1.11), we obtain

$$\frac{|A-a|^2}{4} \leq C |A-a|^2,$$

which gives  $C \geq \frac{1}{4}$ , and the theorem is completely proved. ■

### 1.2.3 Applications for Isotonic Linear Functionals

Let  $F(T)$  be an algebra of real functions defined on  $T$  and  $L$  a subclass of  $F(T)$  satisfying the conditions:

- (i)  $f, g \in L$  implies  $f + g \in L$ ;
- (ii)  $f \in L$ ,  $\alpha \in \mathbb{R}$  implies  $\alpha f \in L$ .

A functional  $A$  defined on  $L$  is an *isotonic linear functional* on  $L$  provided that

- (a)  $A(\alpha f + \beta g) = \alpha A(f) + \beta A(g)$  for all  $\alpha, \beta \in \mathbb{R}$  and  $f, g \in L$ ;
- (aa)  $f \geq g$ , that is,  $f(t) \geq g(t)$  for all  $t \in T$ , implies  $A(f) \geq A(g)$ .

The functional  $A$  is *normalised* on  $L$ , provided that  $\mathbf{1} \in L$ , i.e.,  $\mathbf{1}(t) = 1$  for all  $t \in T$ , implies  $A(\mathbf{1}) = 1$ .

Usual examples of isotonic linear functionals are integrals, sums, etc.

Now, suppose that  $h \in F(T)$ ,  $h \geq 0$  is given and satisfies the properties that  $fgh \in L$ ,  $fh \in L$ ,  $gh \in L$  for all  $f, g \in L$ . For a given isotonic linear functional  $A : L \rightarrow \mathbb{R}$  with  $A(h) > 0$ , define the mapping  $(\cdot, \cdot)_{A,h} : L \times L \rightarrow \mathbb{R}$  by

$$(f, g)_{A,h} := \frac{A(fgh)}{A(h)}.$$

This functional satisfies the following properties:

- (s)  $(f, f)_{A,h} \geq 0$  for all  $f \in L$ ;
- (ss)  $(\alpha f + \beta g, k)_{A,h} = \alpha (f, k)_{A,h} + \beta (g, k)_{A,h}$  for all  $f, g, k \in L$  and  $\alpha, \beta \in \mathbb{R}$ ;
- (sss)  $(f, g)_{A,h} = (g, f)_{A,h}$  for all  $f, g \in L$ .

The following reverse of Schwarz's inequality for positive linear functionals holds [4].

**Proposition 2** *Let  $f, g, h \in F(T)$  be such that  $fgh \in L$ ,  $f^2h \in L$ ,  $g^2h \in L$ . If  $m, M$  are real numbers such that*

$$mg \leq f \leq Mg \text{ on } F(T), \quad (1.12)$$

*then for any isotonic linear functional  $A : L \rightarrow \mathbb{R}$  with  $A(h) > 0$  we have the inequality*

$$0 \leq A(hf^2)A(hg^2) - [A(hfg)]^2 \leq \frac{1}{4}(M-m)^2 A^2(hg^2). \quad (1.13)$$

*The constant  $\frac{1}{4}$  in (1.13) is sharp.*

**Proof.** We observe that

$$(Mg - f, f - mg)_{A,h} = A[h(Mg - f)(f - mg)] \geq 0.$$

Applying Theorem 1 for  $(\cdot, \cdot)_{A,h}$ , we get

$$0 \leq (f, f)_{A,h}(g, g)_{A,h} - (f, g)_{A,h}^2 \leq \frac{1}{4}(M-m)^2 (g, g)_{A,h}^2,$$

which is clearly equivalent to (1.13). ■

The following corollary holds.

**Corollary 3** *Let  $f, g \in F(T)$  such that  $fg, f^2, g^2 \in F(T)$ . If  $m, M$  are real numbers such that (1.12) holds, then*

$$0 \leq A(f^2)A(g^2) - A^2(fg) \leq \frac{1}{4}(M-m)^2 A^2(g^2). \quad (1.14)$$

*The constant  $\frac{1}{4}$  is sharp in (1.14).*

**Remark 4** *The condition (1.12) may be replaced with the weaker assumption*

$$(Mg - f, f - mg)_{A,h} \geq 0.$$

### 1.2.4 Applications for Integrals

Let  $(\Omega, \Sigma, \mu)$  be a measure space consisting of a set  $\Omega$ ,  $\Sigma$  a  $\sigma$ -algebra of subsets of  $\Omega$  and  $\mu$  a countably additive and positive measure on  $\Sigma$  with values in  $\mathbb{R} \cup \{\infty\}$ .

Denote by  $L_\rho^2(\Omega, \mathbb{K})$  the Hilbert space of all  $\mathbb{K}$ -valued functions  $f$  defined on  $\Omega$  that are  $2 - \rho$ -integrable on  $\Omega$ , i.e.,  $\int_\Omega \rho(t) |f(s)|^2 d\mu(s) < \infty$ , where  $\rho : \Omega \rightarrow [0, \infty)$  is a measurable function on  $\Omega$ .

The following proposition contains a reverse of the weighted Cauchy-Bunyakovsky-Schwarz's integral inequality [4].

**Proposition 5** *Let  $A, a \in \mathbb{K}$  ( $\mathbb{K} = \mathbb{C}, \mathbb{R}$ ) and  $f, g \in L_\rho^2(\Omega, \mathbb{K})$ . If*

$$\int_\Omega \operatorname{Re} \left[ (Ag(s) - f(s)) \left( \overline{f(s)} - \bar{a} \overline{g(s)} \right) \right] \rho(s) d\mu(s) \geq 0 \quad (1.15)$$

*or equivalently,*

$$\int_\Omega \rho(s) \left| f(s) - \frac{a+A}{2} g(s) \right|^2 d\mu(s) \leq \frac{1}{4} |A-a|^2 \int_\Omega \rho(s) |g(s)|^2 d\mu(s),$$

*holds, then one has the inequality*

$$\begin{aligned} 0 &\leq \int_\Omega \rho(s) |f(s)|^2 d\mu(s) \int_\Omega \rho(s) |g(s)|^2 d\mu(s) - \left| \int_\Omega \rho(s) f(s) \overline{g(s)} d\mu(s) \right|^2 \\ &\leq \frac{1}{4} |A-a|^2 \left( \int_\Omega \rho(s) |g(s)|^2 d\mu(s) \right)^2. \end{aligned}$$

*The constant  $\frac{1}{4}$  is best possible.*

**Proof.** Follows by Theorem 1 applied for the inner product  $\langle \cdot, \cdot \rangle_\rho := L_\rho^2(\Omega, \mathbb{K}) \times L_\rho^2(\Omega, \mathbb{K}) \rightarrow \mathbb{K}$ ,

$$\langle f, g \rangle_\rho := \int_\Omega \rho(s) f(s) \overline{g(s)} d\mu(s).$$

■

**Remark 6** *A sufficient condition for (1.15) to hold is*

$$\operatorname{Re} \left[ (Ag(s) - f(s)) \left( \overline{f(s)} - \bar{a} \overline{g(s)} \right) \right] \geq 0, \quad \text{for } \mu - a.e. \ s \in \Omega.$$

In the particular case  $\rho = 1$ , we have the following reverse of the Cauchy-Bunyakovsky-Schwarz inequality.

**Corollary 7** *Let  $a, A \in \mathbb{K}$  ( $\mathbb{K} = \mathbb{C}, \mathbb{R}$ ) and  $f, g \in L^2(\Omega, \mathbb{K})$ . If*

$$\int_\Omega \operatorname{Re} \left[ (Ag(s) - f(s)) \left( \overline{f(s)} - \bar{a} \overline{g(s)} \right) \right] d\mu(s) \geq 0, \quad (1.16)$$

or equivalently,

$$\int_{\Omega} \left| f(s) - \frac{a+A}{2} g(s) \right|^2 d\mu(s) \leq \frac{1}{4} |A-a|^2 \int_{\Omega} |g(s)|^2 d\mu(s),$$

holds, then one has the inequality

$$\begin{aligned} 0 &\leq \int_{\Omega} |f(s)|^2 d\mu(s) \int_{\Omega} |g(s)|^2 d\mu(s) - \left| \int_{\Omega} f(s) \overline{g(s)} d\mu(s) \right|^2 \\ &\leq \frac{1}{4} |A-a|^2 \left( \int_{\Omega} |g(s)|^2 d\mu(s) \right)^2. \end{aligned}$$

The constant  $\frac{1}{4}$  is best possible

**Remark 8** If  $\mathbb{K} = \mathbb{R}$ , then a sufficient condition for either (1.15) or (1.16) to hold is

$$ag(s) \leq f(s) \leq Ag(s), \quad \text{for } \mu - a.e. \ s \in \Omega,$$

where, in this case,  $a, A \in \mathbb{R}$  with  $A > a > 0$ .

### 1.2.5 Applications for Sequences

For a given sequence  $(w_i)_{i \in \mathbb{N}}$  of nonnegative real numbers, consider the Hilbert space  $\ell_w^2(\mathbb{K})$ , ( $\mathbb{K} = \mathbb{C}, \mathbb{R}$ ), where

$$\ell_w^2(\mathbb{K}) := \left\{ \bar{\mathbf{x}} = (x_i)_{i \in \mathbb{N}} \subset \mathbb{K} \mid \sum_{i=0}^{\infty} w_i |x_i|^2 < \infty \right\}.$$

The following proposition that provides a reverse of the weighted Cauchy-Bunyakovsky-Schwarz inequality for complex numbers holds.

**Proposition 9** Let  $a, A \in \mathbb{K}$  and  $\bar{\mathbf{x}}, \bar{\mathbf{y}} \in \ell_w^2(\mathbb{K})$ . If

$$\sum_{i=0}^{\infty} w_i \operatorname{Re}[(Ay_i - x_i)(\bar{x}_i - \bar{a} \bar{y}_i)] \geq 0, \quad (1.17)$$

then one has the inequality

$$0 \leq \sum_{i=0}^{\infty} w_i |x_i|^2 \sum_{i=0}^{\infty} w_i |y_i|^2 - \left| \sum_{i=0}^{\infty} w_i x_i \bar{y}_i \right|^2 \leq \frac{1}{4} |A-a|^2 \left( \sum_{i=0}^{\infty} w_i |y_i|^2 \right)^2.$$

The constant  $\frac{1}{4}$  is sharp.

**Proof.** Follows by Theorem 1 applied for the inner product  $\langle \cdot, \cdot \rangle_w : \ell_w^2(\mathbb{K}) \times \ell_w^2(\mathbb{K}) \rightarrow \mathbb{K}$ ,

$$\langle \bar{\mathbf{x}}, \bar{\mathbf{y}} \rangle_w := \sum_{i=0}^{\infty} w_i x_i \bar{y}_i.$$

■



**Remark 10** A sufficient condition for (1.17) to hold is

$$\operatorname{Re}[(Ay_i - x_i)(\overline{x_i} - \overline{ay_i})] \geq 0, \quad \text{for all } i \in \mathbb{N}.$$

In the particular case  $w_i = 1$ ,  $i \in \mathbb{N}$ , we have the following reverse of the Cauchy-Bunyakovsky-Schwarz inequality.

**Corollary 11** Let  $a, A \in \mathbb{K}$  ( $\mathbb{K} = \mathbb{C}, \mathbb{R}$ ) and  $\overline{\mathbf{x}}, \overline{\mathbf{y}} \in \ell^2(\mathbb{K})$ . If

$$\sum_{i=0}^{\infty} \operatorname{Re}[(Ay_i - x_i)(\overline{x_i} - \overline{ay_i})] \geq 0, \quad (1.18)$$

then one has the inequality

$$0 \leq \sum_{i=0}^{\infty} |x_i|^2 \sum_{i=0}^{\infty} |y_i|^2 - \left| \sum_{i=0}^{\infty} x_i \overline{y_i} \right|^2 \leq \frac{1}{4} |A - a|^2 \left( \sum_{i=0}^{\infty} |y_i|^2 \right)^2.$$

**Remark 12** If  $\mathbb{K} = \mathbb{R}$ , then a sufficient condition for either (1.17) or (1.18) to hold is

$$ay_i \leq x_i \leq Ay_i \quad \text{for each } i \in \mathbb{N},$$

with  $A > a > 0$ .

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## 1.3 A Generalisation of the Cassels and Greub-Reinboldt Inequalities

### 1.3.1 Introduction

The following result was proved by J.W.S. Cassels in 1951 (see Appendix 1 of [11]).

**Theorem 13** *Let  $\bar{\mathbf{a}} = (a_1, \dots, a_n)$ ,  $\bar{\mathbf{b}} = (b_1, \dots, b_n)$  be sequences of positive real numbers and  $\bar{\mathbf{w}} = (w_1, \dots, w_n)$  a sequence of nonnegative real numbers. Suppose that*

$$m = \min_{i=1, n} \left\{ \frac{a_i}{b_i} \right\} \quad \text{and} \quad M = \max_{i=1, n} \left\{ \frac{a_i}{b_i} \right\}. \quad (1.19)$$

*Then one has the inequality*

$$\frac{\sum_{i=1}^n w_i a_i^2 \sum_{i=1}^n w_i b_i^2}{\left( \sum_{i=1}^n w_i a_i b_i \right)^2} \leq \frac{(m + M)^2}{4mM}. \quad (1.20)$$

*The equality holds in (1.20) when  $w_1 = \frac{1}{a_1 b_1}$ ,  $w_n = \frac{1}{a_n b_n}$ ,  $w_2 = \dots = w_{n-1} = 0$ ,  $m = \frac{a_1}{b_1}$  and  $M = \frac{a_n}{b_n}$ .*

If one assumes that  $0 < a \leq a_i \leq A < \infty$  and  $0 < b \leq b_i \leq B < \infty$  for each  $i \in \{1, \dots, n\}$ , then by (1.20) we may obtain *Greub-Reinboldt's inequality* [4]

$$\frac{\sum_{i=1}^n w_i a_i^2 \sum_{i=1}^n w_i b_i^2}{\left( \sum_{i=1}^n w_i a_i b_i \right)^2} \leq \frac{(ab + AB)^2}{4abAB}.$$

The following “unweighted” Cassels’ inequality also holds

$$\frac{\sum_{i=1}^n a_i^2 \sum_{i=1}^n b_i^2}{\left( \sum_{i=1}^n a_i b_i \right)^2} \leq \frac{(m + M)^2}{4mM},$$

provided  $\bar{\mathbf{a}}$  and  $\bar{\mathbf{b}}$  satisfy (1.19). This inequality will produce the well known *Pólya-Szegö inequality* [8, pp. 57, 213-214], [5, pp. 71-72, 253-255]:

$$\frac{\sum_{i=1}^n a_i^2 \sum_{i=1}^n b_i^2}{\left( \sum_{i=1}^n a_i b_i \right)^2} \leq \frac{(ab + AB)^2}{4abAB},$$

provided  $0 < a \leq a_i \leq A < \infty$  and  $0 < b \leq b_i \leq B < \infty$  for each  $i \in \{1, \dots, n\}$ .

In [7], C.P. Niculescu proved, amongst others, the following generalisation of Cassels’ inequality:

**Theorem 14** *Let  $E$  be a vector space endowed with a Hermitian product  $\langle \cdot, \cdot \rangle$ . Then*

$$\frac{\operatorname{Re} \langle x, y \rangle}{\langle x, x \rangle^{\frac{1}{2}} \langle y, y \rangle^{\frac{1}{2}}} \geq \frac{2}{\sqrt{\frac{\omega}{\Omega}} + \sqrt{\frac{\Omega}{\omega}}} \quad (1.21)$$

*for every  $x, y \in E$  and every  $\omega, \Omega > 0$  for which  $\operatorname{Re} \langle x - \omega y, x - \Omega y \rangle \leq 0$ .*

For other reverses of the Cauchy-Bunyakovsky-Schwarz inequality, see the references [1] – [11].

In this section, by following [3], we establish a generalisation of (1.21) for complex numbers  $\omega$  and  $\Omega$  for which  $\operatorname{Re}(\overline{\omega}\Omega) > 0$ . Applications for isotonic linear functionals, integrals and sequences are also given.

### 1.3.2 An Inequality in Real or Complex Inner Product Spaces

The following reverse of Schwarz's inequality in inner product spaces holds [3].

**Theorem 15** *Let  $a, A \in \mathbb{K}$  ( $\mathbb{K} = \mathbb{C}, \mathbb{R}$ ) so that  $\operatorname{Re}(\overline{a}A) > 0$ . If  $x, y \in H$  are such that*

$$\operatorname{Re} \langle Ay - x, x - ay \rangle \geq 0, \quad (1.22)$$

*then one has the inequality*

$$\|x\| \|y\| \leq \frac{1}{2} \cdot \frac{\operatorname{Re} [A\overline{\langle x, y \rangle} + \overline{a} \langle x, y \rangle]}{[\operatorname{Re}(\overline{a}A)]^{\frac{1}{2}}} \leq \frac{1}{2} \cdot \frac{|A| + |a|}{[\operatorname{Re}(\overline{a}A)]^{\frac{1}{2}}} |\langle x, y \rangle|. \quad (1.23)$$

*The constant  $\frac{1}{2}$  is sharp in both inequalities.*

**Proof.** We have, obviously, that

$$\begin{aligned} I &:= \operatorname{Re} \langle Ay - x, x - ay \rangle \\ &= \operatorname{Re} [A\overline{\langle x, y \rangle} + \overline{a} \langle x, y \rangle] - \|x\|^2 - [\operatorname{Re}(\overline{a}A)] \|y\|^2 \end{aligned}$$

and, thus, by (1.22), one has

$$\|x\|^2 + [\operatorname{Re}(\overline{a}A)] \|y\|^2 \leq \operatorname{Re} [A\overline{\langle x, y \rangle} + \overline{a} \langle x, y \rangle],$$

which gives

$$\frac{1}{[\operatorname{Re}(\overline{a}A)]^{\frac{1}{2}}} \|x\|^2 + [\operatorname{Re}(\overline{a}A)]^{\frac{1}{2}} \|y\|^2 \leq \frac{\operatorname{Re} [A\overline{\langle x, y \rangle} + \overline{a} \langle x, y \rangle]}{[\operatorname{Re}(\overline{a}A)]^{\frac{1}{2}}}. \quad (1.24)$$

On the other hand, by the elementary inequality

$$\alpha p^2 + \frac{1}{\alpha} q^2 \geq 2pq,$$

valid for  $p, q \geq 0$  and  $\alpha > 0$ , we deduce

$$2 \|x\| \|y\| \leq \frac{1}{[\operatorname{Re}(\overline{a}A)]^{\frac{1}{2}}} \|x\|^2 + [\operatorname{Re}(\overline{a}A)]^{\frac{1}{2}} \|y\|^2. \quad (1.25)$$

Utilizing (1.24) and (1.25) we deduce the first part of (1.23).

The second part is obvious by the fact that for  $z \in \mathbb{C}$ ,  $|\operatorname{Re}(z)| \leq |z|$ .

Now, assume that the first inequality in (1.23) holds with a constant  $c > 0$ , i.e.,

$$\|x\| \|y\| \leq c \frac{\operatorname{Re} [A \overline{\langle x, y \rangle} + \bar{a} \langle x, y \rangle]}{[\operatorname{Re}(\bar{a}A)]^{\frac{1}{2}}}, \quad (1.26)$$

where  $a, A, x$  and  $y$  satisfy (1.23).

If we choose  $a = A = 1$ ,  $y = x \neq 0$ , then obviously (1.22) holds and from (1.26) we obtain

$$\|x\|^2 \leq 2c \|x\|^2,$$

giving  $c \geq \frac{1}{2}$ .

The theorem is completely proved. ■

The following corollary is a natural consequence of the above theorem [3].

**Corollary 16** *Let  $m, M > 0$ . If  $x, y \in H$  are such that*

$$\operatorname{Re} \langle My - x, x - my \rangle \geq 0,$$

*then one has the inequality*

$$\|x\| \|y\| \leq \frac{1}{2} \cdot \frac{M+m}{\sqrt{mM}} \operatorname{Re} \langle x, y \rangle \leq \frac{1}{2} \cdot \frac{M+m}{\sqrt{mM}} |\langle x, y \rangle|. \quad (1.27)$$

*The constant  $\frac{1}{2}$  is sharp in (1.27).*

**Remark 17** *The inequality (1.27) is equivalent to Niculescu's inequality (1.21).*

The following corollary is also obvious [3].

**Corollary 18** *With the assumptions of Corollary 16, we have*

$$\begin{aligned} 0 &\leq \|x\| \|y\| - |\langle x, y \rangle| \leq \|x\| \|y\| - \operatorname{Re} \langle x, y \rangle \\ &\leq \frac{(\sqrt{M} - \sqrt{m})^2}{2\sqrt{mM}} \operatorname{Re} \langle x, y \rangle \leq \frac{(\sqrt{M} - \sqrt{m})^2}{2\sqrt{mM}} |\langle x, y \rangle| \end{aligned} \quad (1.28)$$

*and*

$$\begin{aligned} 0 &\leq \|x\|^2 \|y\|^2 - |\langle x, y \rangle|^2 \leq \|x\|^2 \|y\|^2 - [\operatorname{Re} \langle x, y \rangle]^2 \\ &\leq \frac{(M-m)^2}{4mM} [\operatorname{Re} \langle x, y \rangle]^2 \leq \frac{(M-m)^2}{4mM} |\langle x, y \rangle|^2. \end{aligned} \quad (1.29)$$

*The constants  $\frac{1}{2}$  and  $\frac{1}{4}$  are sharp.*

**Proof.** If we subtract  $\operatorname{Re} \langle x, y \rangle \geq 0$  from the first inequality in (1.27), we get

$$\begin{aligned} \|x\| \|y\| - \operatorname{Re} \langle x, y \rangle &\leq \left( \frac{1}{2} \cdot \frac{M+m}{\sqrt{mM}} - 1 \right) \operatorname{Re} \langle x, y \rangle \\ &= \frac{(\sqrt{M} - \sqrt{m})^2}{2\sqrt{mM}} \operatorname{Re} \langle x, y \rangle \end{aligned}$$

which proves the third inequality in (1.28). The other ones are obvious.

Now, if we square the first inequality in (1.27) and then subtract  $[\operatorname{Re} \langle x, y \rangle]^2$ , we get

$$\begin{aligned} \|x\|^2 \|y\|^2 - [\operatorname{Re} \langle x, y \rangle]^2 &\leq \left[ \frac{(M+m)^2}{4mM} - 1 \right] [\operatorname{Re} \langle x, y \rangle]^2 \\ &= \frac{(M-m)^2}{4mM} [\operatorname{Re} \langle x, y \rangle]^2 \end{aligned}$$

which proves the third inequality in (1.29). The other ones are obvious. ■

### 1.3.3 Applications for Isotonic Linear Functionals

The following proposition holds [3].

**Proposition 19** *Let  $f, g, h \in F(T)$  be such that  $fgh \in L$ ,  $f^2h \in L$ ,  $g^2h \in L$ . If  $m, M > 0$  are such that*

$$mg \leq f \leq Mg \text{ on } F(T), \quad (1.30)$$

*then for any isotonic linear functional  $A : L \rightarrow \mathbb{R}$  with  $A(h) > 0$ , we have the inequality*

$$1 \leq \frac{A(f^2h) A(g^2h)}{A^2(fgh)} \leq \frac{(M+m)^2}{4mM}. \quad (1.31)$$

*The constant  $\frac{1}{4}$  in (1.31) is sharp.*

**Proof.** We observe that

$$(Mg - f, f - mg)_{A,h} = A[h(Mg - f)(f - mg)] \geq 0.$$

Applying Corollary 16 for  $(\cdot, \cdot)_{A,h}$  we get

$$1 \leq \frac{(f, f)_{A,h} (g, g)_{A,h}}{(f, g)_{A,h}^2} \leq \frac{(M+m)^2}{4mM},$$

which is clearly equivalent to (1.31). ■

The following additive versions of (1.31) also hold [3].

**Corollary 20** *With the assumption in Proposition 19, one has*

$$\begin{aligned} 0 &\leq [A(f^2 h) A(g^2 h)]^{\frac{1}{2}} - A(hfg) \\ &\leq \frac{(\sqrt{M} - \sqrt{m})^2}{2\sqrt{mM}} A(hfg) \end{aligned}$$

and

$$\begin{aligned} 0 &\leq A(f^2 h) A(g^2 h) - A^2(fgh) \\ &\leq \frac{(M - m)^2}{4mM} A^2(fgh). \end{aligned}$$

The constants  $\frac{1}{2}$  and  $\frac{1}{4}$  are sharp.

**Remark 21** *The condition (1.30) may be replaced with the weaker assumption*

$$(Mg - f, f - mg)_{A,h} \geq 0. \quad (1.32)$$

**Remark 22** *With the assumption (1.30) or (1.32) and if  $f, g \in F(T)$  with  $fg, f^2, g^2 \in L$ , then one has the inequalities*

$$1 \leq \frac{A(f^2) A(g^2)}{A^2(fg)} \leq \frac{(M + m)^2}{4mM},$$

$$\begin{aligned} 0 &\leq [A(f^2) A(g^2)]^{\frac{1}{2}} - A(fg) \\ &\leq \frac{(\sqrt{M} - \sqrt{m})^2}{2\sqrt{mM}} A(fg) \end{aligned}$$

and

$$0 \leq A(f^2) A(g^2) - A^2(fg) \leq \frac{(M - m)^2}{4mM} A^2(fg).$$

### 1.3.4 Applications for Integrals

The following proposition contains a reverse of the weighted Cauchy-Bunyakovsky-Schwarz integral inequality.

**Proposition 23** *Let  $A, a \in \mathbb{K}$  ( $\mathbb{K} = \mathbb{C}, \mathbb{R}$ ) with  $\operatorname{Re}(\bar{a}A) > 0$  and  $f, g \in L_\rho^2(\Omega, \mathbb{K})$ . If*

$$\int_{\Omega} \operatorname{Re} \left[ (Ag(s) - f(s)) \left( \overline{f(s)} - \bar{a} \bar{g}(s) \right) \right] \rho(s) d\mu(s) \geq 0, \quad (1.33)$$

then one has the inequality

$$\begin{aligned}
& \left[ \int_{\Omega} \rho(s) |f(s)|^2 d\mu(s) \int_{\Omega} \rho(s) |g(s)|^2 d\mu(s) \right]^{\frac{1}{2}} \\
& \leq \frac{1}{2} \cdot \frac{\int_{\Omega} \rho(s) \operatorname{Re} \left[ A \overline{f(s)} g(s) + \overline{a} f(s) \overline{g(s)} \right] d\mu(s)}{[\operatorname{Re}(\overline{a}A)]^{\frac{1}{2}}} \\
& \leq \frac{1}{2} \cdot \frac{|A| + |a|}{[\operatorname{Re}(\overline{a}A)]^{\frac{1}{2}}} \left| \int_{\Omega} \rho(s) f(s) \overline{g(s)} d\mu(s) \right|.
\end{aligned} \tag{1.34}$$

The constant  $\frac{1}{2}$  is sharp in (1.34).

**Proof.** Follows by Theorem 15 applied for the inner product  $\langle \cdot, \cdot \rangle_{\rho} := L_{\rho}^2(\Omega, \mathbb{K}) \times L_{\rho}^2(\Omega, \mathbb{K}) \rightarrow \mathbb{K}$ ,

$$\langle f, g \rangle := \int_{\Omega} \rho(s) f(s) \overline{g(s)} d\mu(s).$$

■

**Remark 24** A sufficient condition for (1.33) to hold is

$$\operatorname{Re} \left[ (Ag(s) - f(s)) \left( \overline{f(s)} - \overline{a} \overline{g(s)} \right) \right] \geq 0, \quad \text{for } \mu\text{-a.e. } s \in \Omega.$$

In the particular case  $\rho = 1$ , we have the following result.

**Corollary 25** Let  $a, A \in \mathbb{K}$  ( $\mathbb{K} = \mathbb{C}, \mathbb{R}$ ) with  $\operatorname{Re}(\overline{a}A) > 0$  and  $f, g \in L^2(\Omega, \mathbb{K})$ . If

$$\int_{\Omega} \operatorname{Re} \left[ (Ag(s) - f(s)) \left( \overline{f(s)} - \overline{a} \overline{g(s)} \right) \right] d\mu(s) \geq 0, \tag{1.35}$$

then one has the inequality

$$\begin{aligned}
& \left[ \int_{\Omega} |f(s)|^2 d\mu(s) \int_{\Omega} |g(s)|^2 d\mu(s) \right]^{\frac{1}{2}} \\
& \leq \frac{1}{2} \cdot \frac{\int_{\Omega} \operatorname{Re} \left[ A \overline{f(s)} g(s) + \overline{a} f(s) \overline{g(s)} \right] d\mu(s)}{[\operatorname{Re}(\overline{a}A)]^{\frac{1}{2}}} \\
& \leq \frac{1}{2} \cdot \frac{|A| + |a|}{[\operatorname{Re}(\overline{a}A)]^{\frac{1}{2}}} \left| \int_{\Omega} f(s) \overline{g(s)} d\mu(s) \right|.
\end{aligned}$$

**Remark 26** If  $\mathbb{K} = \mathbb{R}$ , then a sufficient condition for either (1.33) or (1.35) to hold is

$$ag(s) \leq f(s) \leq Ag(s), \quad \text{for } \mu\text{-a.e. } s \in \Omega,$$

where, in this case,  $a, A \in \mathbb{R}$  with  $A > a > 0$ .

If  $a, A$  are real positive constants, then the following proposition holds.



**Proposition 27** *Let  $m, M > 0$ . If  $f, g \in L^2_\rho(\Omega, \mathbb{K})$  such that*

$$\int_{\Omega} \rho(s) \operatorname{Re} \left[ (Mg(s) - f(s)) \left( \overline{f(s)} - m\overline{g(s)} \right) \right] d\mu(s) \geq 0,$$

*then one has the inequality*

$$\begin{aligned} & \left[ \int_{\Omega} \rho(s) |f(s)|^2 d\mu(s) \int_{\Omega} \rho(s) |g(s)|^2 d\mu(s) \right]^{\frac{1}{2}} \\ & \leq \frac{1}{2} \cdot \frac{M+m}{\sqrt{mM}} \int_{\Omega} \rho(s) \operatorname{Re} \left[ f(s) \overline{g(s)} \right] d\mu(s). \end{aligned}$$

The proof follows by Corollary 16 applied for the inner product

$$\langle f, g \rangle_{\rho} := \int_{\Omega} \rho(s) f(s) \overline{g(s)} d\mu(s).$$

The following additive versions also hold [3].

**Corollary 28** *With the assumptions in Proposition 27, one has the inequalities*

$$\begin{aligned} 0 & \leq \left[ \int_{\Omega} \rho(s) |f(s)|^2 d\mu(s) \int_{\Omega} \rho(s) |g(s)|^2 d\mu(s) \right]^{\frac{1}{2}} \\ & \quad - \int_{\Omega} \rho(s) \operatorname{Re} \left[ f(s) \overline{g(s)} \right] d\mu(s) \\ & \leq \frac{(\sqrt{M} - \sqrt{m})^2}{2\sqrt{mM}} \int_{\Omega} \rho(s) \operatorname{Re} \left[ f(s) \overline{g(s)} \right] d\mu(s) \end{aligned}$$

and

$$\begin{aligned} 0 & \leq \int_{\Omega} \rho(s) |f(s)|^2 d\mu(s) \int_{\Omega} \rho(s) |g(s)|^2 d\mu(s) \\ & \quad - \left( \int_{\Omega} \rho(s) \operatorname{Re} \left[ f(s) \overline{g(s)} \right] d\mu(s) \right)^2 \\ & \leq \frac{(M-m)^2}{4mM} \left( \int_{\Omega} \rho(s) \operatorname{Re} \left[ f(s) \overline{g(s)} \right] d\mu(s) \right)^2. \end{aligned}$$

**Remark 29** *If  $\mathbb{K} = \mathbb{R}$ , a sufficient condition for (1.33) to hold is*

$$mg(s) \leq f(s) \leq Mg(s), \quad \text{for } \mu\text{-a.e. } s \in \Omega,$$

*where  $M > m > 0$ .*

### 1.3.5 Applications for Sequences

For a given sequence  $(w_i)_{i \in \mathbb{N}}$  of nonnegative real numbers, consider the Hilbert space  $\ell_w^2(\mathbb{K})$ , ( $\mathbb{K} = \mathbb{C}, \mathbb{R}$ ), where

$$\ell_w^2(\mathbb{K}) := \left\{ \bar{\mathbf{x}} = (x_i)_{i \in \mathbb{N}} \subset \mathbb{K} \left| \sum_{i=0}^{\infty} w_i |x_i|^2 < \infty \right. \right\}.$$

The following proposition that provides a reverse of the weighted Cauchy-Bunyakovsky-Schwarz inequality for complex numbers holds [3].

**Proposition 30** *Let  $a, A \in \mathbb{K}$  with  $\operatorname{Re}(\bar{a}A) > 0$  and  $\bar{\mathbf{x}}, \bar{\mathbf{y}} \in \ell_w^2(\mathbb{K})$ . If*

$$\sum_{i=0}^{\infty} w_i \operatorname{Re}[(Ay_i - x_i)(\bar{x}_i - \bar{a}\bar{y}_i)] \geq 0, \quad (1.36)$$

*then one has the inequality*

$$\begin{aligned} \left[ \sum_{i=0}^{\infty} w_i |x_i|^2 \sum_{i=0}^{\infty} w_i |y_i|^2 \right]^{\frac{1}{2}} &\leq \frac{1}{2} \cdot \frac{\sum_{i=0}^{\infty} w_i \operatorname{Re}[A\bar{x}_i y_i + \bar{a}x_i \bar{y}_i]}{[\operatorname{Re}(\bar{a}A)]^{\frac{1}{2}}} \\ &\leq \frac{1}{2} \cdot \frac{|A| + |a|}{[\operatorname{Re}(\bar{a}A)]^{\frac{1}{2}}} \left| \sum_{i=0}^{\infty} w_i x_i \bar{y}_i \right|. \end{aligned} \quad (1.37)$$

*The constant  $\frac{1}{2}$  is sharp in (1.37).*

**Proof.** Follows by Theorem 15 applied for the inner product  $\langle \cdot, \cdot \rangle : \ell_w^2(\mathbb{K}) \times \ell_w^2(\mathbb{K}) \rightarrow \mathbb{K}$ ,

$$\langle \bar{\mathbf{x}}, \bar{\mathbf{y}} \rangle_w := \sum_{i=0}^{\infty} w_i x_i \bar{y}_i.$$

■

**Remark 31** *A sufficient condition for (1.36) to hold is*

$$\operatorname{Re}[(Ay_i - x_i)(\bar{x}_i - \bar{a}\bar{y}_i)] \geq 0 \quad \text{for all } i \in \mathbb{N}. \quad (1.38)$$

In the particular case  $\rho = 1$ , we have the following result.

**Corollary 32** *Let  $a, A \in \mathbb{K}$  with  $\operatorname{Re}(\bar{a}A) > 0$  and  $\bar{\mathbf{x}}, \bar{\mathbf{y}} \in \ell^2(\mathbb{K})$ . If*

$$\sum_{i=0}^{\infty} \operatorname{Re}[(Ay_i - x_i)(\bar{x}_i - \bar{a}\bar{y}_i)] \geq 0,$$

*then one has the inequality*

$$\begin{aligned} \left[ \sum_{i=0}^{\infty} |x_i|^2 \sum_{i=0}^{\infty} |y_i|^2 \right]^{\frac{1}{2}} &\leq \frac{1}{2} \cdot \frac{\sum_{i=0}^{\infty} \operatorname{Re}[A\bar{x}_i y_i + \bar{a}x_i \bar{y}_i]}{[\operatorname{Re}(\bar{a}A)]^{\frac{1}{2}}} \\ &\leq \frac{1}{2} \cdot \frac{|A| + |a|}{[\operatorname{Re}(\bar{a}A)]^{\frac{1}{2}}} \left| \sum_{i=0}^{\infty} x_i \bar{y}_i \right|. \end{aligned}$$

**Remark 33** If  $\mathbb{K} = \mathbb{R}$ , then a sufficient condition for either (1.36) or (1.38) to hold is

$$ay_i \leq x_i \leq Ay_i \text{ for each } i \in \{1, \dots, n\},$$

where, in this case,  $a, A \in \mathbb{R}$  with  $A > a > 0$ .

For  $a = m$ ,  $A = M$ , then the following proposition also holds.

**Proposition 34** Let  $m, M > 0$ . If  $\bar{\mathbf{x}}, \bar{\mathbf{y}} \in \ell_w^2(\mathbb{K})$  such that

$$\sum_{i=0}^{\infty} w_i \operatorname{Re} [(My_i - x_i) (\bar{x}_i - m\bar{y}_i)] \geq 0, \quad (1.39)$$

then one has the inequality

$$\left[ \sum_{i=0}^{\infty} w_i |x_i|^2 \sum_{i=0}^{\infty} w_i |y_i|^2 \right]^{\frac{1}{2}} \leq \frac{1}{2} \cdot \frac{M+m}{\sqrt{mM}} \sum_{i=0}^{\infty} w_i \operatorname{Re} (x_i \bar{y}_i).$$

The proof follows by Corollary 16 applied for the inner product

$$\langle \bar{\mathbf{x}}, \bar{\mathbf{y}} \rangle_w := \sum_{i=0}^{\infty} w_i x_i \bar{y}_i.$$

The following additive version also holds [3].

**Corollary 35** With the assumptions in Proposition 34, one has the inequalities

$$\begin{aligned} 0 &\leq \left[ \sum_{i=0}^{\infty} w_i |x_i|^2 \sum_{i=0}^{\infty} w_i |y_i|^2 \right]^{\frac{1}{2}} - \sum_{i=0}^{\infty} w_i \operatorname{Re} (x_i \bar{y}_i) \\ &\leq \frac{(\sqrt{M} - \sqrt{m})^2}{2\sqrt{mM}} \sum_{i=0}^{\infty} w_i \operatorname{Re} (x_i \bar{y}_i) \end{aligned}$$

and

$$\begin{aligned} 0 &\leq \sum_{i=0}^{\infty} w_i |x_i|^2 \sum_{i=0}^{\infty} w_i |y_i|^2 - \left[ \sum_{i=0}^{\infty} w_i \operatorname{Re} (x_i \bar{y}_i) \right]^2 \\ &\leq \frac{(M-m)^2}{4mM} \left[ \sum_{i=0}^{\infty} w_i \operatorname{Re} (x_i \bar{y}_i) \right]^2. \end{aligned}$$

**Remark 36** If  $\mathbb{K} = \mathbb{R}$ , a sufficient condition for (1.39) to hold is

$$my_i \leq x_i \leq My_i \text{ for each } i \in \mathbb{N},$$

where  $M > m > 0$ .

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## 1.4 Quadratic Reverses of Schwarz's Inequality

### 1.4.1 Two Better Reverse Inequalities

It has been proven in [6], that

$$0 \leq \|x\|^2 - |\langle x, e \rangle|^2 \leq \frac{1}{4} |\phi - \varphi|^2 - \left| \frac{\phi + \varphi}{2} - \langle x, e \rangle \right|^2, \quad (1.40)$$

provided, either

$$\operatorname{Re} \langle \phi e - x, x - \varphi e \rangle \geq 0, \quad (1.41)$$

or equivalently,

$$\left\| x - \frac{\phi + \varphi}{2} e \right\| \leq \frac{1}{2} |\phi - \varphi|, \quad (1.42)$$

holds, where  $e = H$ ,  $\|e\| = 1$ . The constant  $\frac{1}{4}$  in (1.40) is best possible.

If we choose  $e = \frac{y}{\|y\|}$ ,  $\phi = \Gamma \|y\|$ ,  $\varphi = \gamma \|y\|$  ( $y \neq 0$ ),  $\Gamma, \gamma \in \mathbb{K}$ , then by (1.41) and (1.42) we have,

$$\operatorname{Re} \langle \Gamma y - x, x - \gamma y \rangle \geq 0, \quad (1.43)$$

or equivalently,

$$\left\| x - \frac{\Gamma + \gamma}{2} y \right\| \leq \frac{1}{2} |\Gamma - \gamma| \|y\|, \quad (1.44)$$

implying the following reverse of Schwarz's inequality:

$$0 \leq \|x\|^2 \|y\|^2 - |\langle x, y \rangle|^2 \leq \frac{1}{4} |\Gamma - \gamma|^2 \|y\|^4 - \left| \frac{\Gamma + \gamma}{2} \|y\|^2 - \langle x, y \rangle \right|^2. \quad (1.45)$$

The constant  $\frac{1}{4}$  in (1.45) is sharp.

Note that, this inequality is an improvement of (1.7), but it may not be very convenient for applications.

In [8], it has also been proven that

$$0 \leq \|x\|^2 - |\langle x, e \rangle|^2 \leq \frac{1}{4} |\phi - \varphi|^2 - \operatorname{Re} \langle \Phi e - x, x - \varphi e \rangle \quad (1.46)$$

provided either (1.41) or (1.42) holds true.

If we make the same choice for  $e, \Phi$  and  $\varphi$  as above, then we deduce the inequality

$$0 \leq \|x\|^2 \|y\|^2 - |\langle x, y \rangle|^2 \leq \frac{1}{4} |\Gamma - \gamma|^2 \|y\|^4 - \|y\|^2 \operatorname{Re} \langle \Gamma y - x, x - \gamma y \rangle \quad (1.47)$$

provided either (1.43) or (1.44) holds true.

The constant  $\frac{1}{4}$  is best possible in (1.47).

### 1.4.2 A Reverse of Schwarz's Inequality Under More General Assumptions

The following result holds [7].

**Theorem 37** *Let  $(H; \langle \cdot, \cdot \rangle)$  be an inner product space over the real or complex number field  $\mathbb{K}$  ( $\mathbb{K} = \mathbb{R}$ ,  $\mathbb{K} = \mathbb{C}$ ) and  $x, a \in H$ ,  $r > 0$  are such that*

$$x \in \overline{B}(a, r) := \{z \in H \mid \|z - a\| \leq r\}.$$

(i) *If  $\|a\| > r$ , then we have the inequalities*

$$0 \leq \|x\|^2 \|a\|^2 - |\langle x, a \rangle|^2 \leq \|x\|^2 \|a\|^2 - [\operatorname{Re} \langle x, a \rangle]^2 \leq r^2 \|x\|^2. \quad (1.48)$$

*The constant  $C = 1$  in front of  $r^2$  is best possible in the sense that it cannot be replaced by a smaller one.*

(ii) *If  $\|a\| = r$ , then*

$$\|x\|^2 \leq 2 \operatorname{Re} \langle x, a \rangle \leq 2 |\langle x, a \rangle|. \quad (1.49)$$

*The constant 2 is best possible in both inequalities.*

(iii) *If  $\|a\| < r$ , then*

$$\|x\|^2 \leq r^2 - \|a\|^2 + 2 \operatorname{Re} \langle x, a \rangle \leq r^2 - \|a\|^2 + 2 |\langle x, a \rangle|. \quad (1.50)$$

*Here the constant 2 is also best possible.*

**Proof.** Since  $x \in \overline{B}(a, r)$ , then obviously  $\|x - a\|^2 \leq r^2$ , which is equivalent to

$$\|x\|^2 + \|a\|^2 - r^2 \leq 2 \operatorname{Re} \langle x, a \rangle. \quad (1.51)$$

(i) If  $\|a\| > r$ , then we may divide (1.51) by  $\sqrt{\|a\|^2 - r^2} > 0$  getting

$$\frac{\|x\|^2}{\sqrt{\|a\|^2 - r^2}} + \sqrt{\|a\|^2 - r^2} \leq \frac{2 \operatorname{Re} \langle x, a \rangle}{\sqrt{\|a\|^2 - r^2}}. \quad (1.52)$$

Using the elementary inequality

$$\alpha p + \frac{1}{\alpha} q \geq 2\sqrt{pq}, \quad \alpha > 0, \quad p, q \geq 0,$$

we may state that

$$2 \|x\| \leq \frac{\|x\|^2}{\sqrt{\|a\|^2 - r^2}} + \sqrt{\|a\|^2 - r^2}. \quad (1.53)$$

Making use of (1.52) and (1.53), we deduce

$$\|x\| \sqrt{\|a\|^2 - r^2} \leq \operatorname{Re} \langle x, a \rangle, \quad (1.54)$$

which is an interesting inequality in itself as well.

Taking the square in (1.54) and re-arranging the terms, we deduce the third inequality in (1.48). The others are obvious.

To prove the sharpness of the constant, assume, under the hypothesis of the theorem, that, there exists a constant  $c > 0$  such that

$$\|x\|^2 \|a\|^2 - [\operatorname{Re} \langle x, a \rangle]^2 \leq cr^2 \|x\|^2, \quad (1.55)$$

provided  $x \in \overline{B}(a, r)$  and  $\|a\| > r$ .

Let  $r = \sqrt{\varepsilon} > 0$ ,  $\varepsilon \in (0, 1)$ ,  $a, e \in H$  with  $\|a\| = \|e\| = 1$  and  $a \perp e$ . Put  $x = a + \sqrt{\varepsilon}e$ . Then obviously  $x \in \overline{B}(a, r)$ ,  $\|a\| > r$  and  $\|x\|^2 = \|a\|^2 + \varepsilon \|e\|^2 = 1 + \varepsilon$ ,  $\operatorname{Re} \langle x, a \rangle = \|a\|^2 = 1$ , and thus  $\|x\|^2 \|a\|^2 - [\operatorname{Re} \langle x, a \rangle]^2 = \varepsilon$ . Using (1.55), we may write that

$$\varepsilon \leq c\varepsilon (1 + \varepsilon), \quad \varepsilon > 0$$

giving

$$c + c\varepsilon \geq 1 \quad \text{for any } \varepsilon > 0. \quad (1.56)$$

Letting  $\varepsilon \rightarrow 0+$ , we get from (1.56) that  $c \geq 1$ , and the sharpness of the constant is proved.

- (ii) The inequality (1.49) is obvious by (1.51) since  $\|a\| = r$ . The best constant follows in a similar way to the above.
- (iii) The inequality (1.50) is obvious. The best constant may be proved in a similar way to the above. We omit the details.

■

The following reverse of Schwarz's inequality holds [7].

**Theorem 38** *Let  $(H; \langle \cdot, \cdot \rangle)$  be an inner product space over  $\mathbb{K}$  and  $x, y \in H$ ,  $\gamma, \Gamma \in \mathbb{K}$  such that either*

$$\operatorname{Re} \langle \Gamma y - x, x - \gamma y \rangle \geq 0, \quad (1.57)$$

*or equivalently,*

$$\left\| x - \frac{\Gamma + \gamma}{2} y \right\| \leq \frac{1}{2} |\Gamma - \gamma| \|y\|, \quad (1.58)$$

*holds.*

- (i) *If  $\operatorname{Re}(\Gamma \overline{\gamma}) > 0$ , then we have the inequalities*

$$\begin{aligned} \|x\|^2 \|y\|^2 &\leq \frac{1}{4} \cdot \frac{\{\operatorname{Re}[(\overline{\Gamma} + \overline{\gamma}) \langle x, y \rangle]\}^2}{\operatorname{Re}(\Gamma \overline{\gamma})} \\ &\leq \frac{1}{4} \cdot \frac{|\Gamma + \gamma|^2}{\operatorname{Re}(\Gamma \overline{\gamma})} |\langle x, y \rangle|^2. \end{aligned} \quad (1.59)$$

*The constant  $\frac{1}{4}$  is best possible in both inequalities.*

(ii) If  $\operatorname{Re}(\Gamma\bar{\gamma}) = 0$ , then

$$\|x\|^2 \leq \operatorname{Re}[(\bar{\Gamma} + \bar{\gamma})\langle x, y \rangle] \leq |\Gamma + \gamma| |\langle x, y \rangle|.$$

(iii) If  $\operatorname{Re}(\Gamma\bar{\gamma}) < 0$ , then

$$\begin{aligned} \|x\|^2 &\leq -\operatorname{Re}(\Gamma\bar{\gamma})\|y\|^2 + \operatorname{Re}[(\bar{\Gamma} + \bar{\gamma})\langle x, y \rangle] \\ &\leq -\operatorname{Re}(\Gamma\bar{\gamma})\|y\|^2 + |\Gamma + \gamma| |\langle x, y \rangle|. \end{aligned}$$

**Proof.** The proof of the equivalence between the inequalities (1.57) and (1.58) follows by the fact that in an inner product space  $\operatorname{Re}\langle Z - x, x - z \rangle \geq 0$  for  $x, z, Z \in H$  is equivalent with  $\|x - \frac{z+Z}{2}\| \leq \frac{1}{2}\|Z - z\|$  (see for example [5]).

Consider, for  $y \neq 0$ ,  $a = \frac{\gamma+\Gamma}{2}y$  and  $r = \frac{1}{2}|\Gamma - \gamma|\|y\|$ . Then

$$\|a\|^2 - r^2 = \frac{|\Gamma + \gamma|^2 - |\Gamma - \gamma|^2}{4} \|y\|^2 = \operatorname{Re}(\Gamma\bar{\gamma})\|y\|^2.$$

(i) If  $\operatorname{Re}(\Gamma\bar{\gamma}) > 0$ , then the hypothesis of (i) in Theorem 37 is satisfied, and by the second inequality in (1.48) we have

$$\|x\|^2 \frac{|\Gamma + \gamma|^2}{4} \|y\|^2 - \frac{1}{4} \{\operatorname{Re}[(\bar{\Gamma} + \bar{\gamma})\langle x, y \rangle]\}^2 \leq \frac{1}{4} |\Gamma - \gamma|^2 \|x\|^2 \|y\|^2$$

from where we derive

$$\frac{|\Gamma + \gamma|^2 - |\Gamma - \gamma|^2}{4} \|x\|^2 \|y\|^2 \leq \frac{1}{4} \{\operatorname{Re}[(\bar{\Gamma} + \bar{\gamma})\langle x, y \rangle]\}^2,$$

giving the first inequality in (1.59).

The second inequality is obvious.

To prove the sharpness of the constant  $\frac{1}{4}$ , assume that the first inequality in (1.59) holds with a constant  $c > 0$ , i.e.,

$$\|x\|^2 \|y\|^2 \leq c \cdot \frac{\{\operatorname{Re}[(\bar{\Gamma} + \bar{\gamma})\langle x, y \rangle]\}^2}{\operatorname{Re}(\Gamma\bar{\gamma})}, \quad (1.60)$$

provided  $\operatorname{Re}(\Gamma\bar{\gamma}) > 0$  and either (1.57) or (1.58) holds.

Assume that  $\Gamma, \gamma > 0$ , and let  $x = \gamma y$ . Then (1.57) holds and by (1.60) we deduce

$$\gamma^2 \|y\|^4 \leq c \cdot \frac{(\Gamma + \gamma)^2 \gamma^2 \|y\|^4}{\Gamma\gamma}$$

giving

$$\Gamma\gamma \leq c(\Gamma + \gamma)^2 \quad \text{for any } \Gamma, \gamma > 0. \quad (1.61)$$

Let  $\varepsilon \in (0, 1)$  and choose in (1.61),  $\Gamma = 1 + \varepsilon$ ,  $\gamma = 1 - \varepsilon > 0$  to get  $1 - \varepsilon^2 \leq 4c$  for any  $\varepsilon \in (0, 1)$ . Letting  $\varepsilon \rightarrow 0+$ , we deduce  $c \geq \frac{1}{4}$ , and the sharpness of the constant is proved.

(ii) and (iii) are obvious and we omit the details.



■

**Remark 39** We observe that the second bound in (1.59) for  $\|x\|^2 \|y\|^2$  is better than the second bound provided by (1.23).

The following corollary provides a reverse inequality for the additive version of Schwarz's inequality [7].

**Corollary 40** With the assumptions of Theorem 38 and if  $\operatorname{Re}(\Gamma\overline{\gamma}) > 0$ , then we have the inequality:

$$0 \leq \|x\|^2 \|y\|^2 - |\langle x, y \rangle|^2 \leq \frac{1}{4} \cdot \frac{|\Gamma - \gamma|^2}{\operatorname{Re}(\Gamma\overline{\gamma})} |\langle x, y \rangle|^2. \quad (1.62)$$

The constant  $\frac{1}{4}$  is best possible in (1.62).

The proof is obvious from (1.59) on subtracting in both sides the same quantity  $|\langle x, y \rangle|^2$ . The sharpness of the constant may be proven in a similar manner to the one incorporated in the proof of (i), Theorem 38. We omit the details.

For other recent results in connection to Schwarz's inequality, see [1], [9] and [10].

### 1.4.3 Reverses of the Triangle Inequality

The following reverse of the triangle inequality holds [7].

**Proposition 41** Let  $(H; \langle \cdot, \cdot \rangle)$  be an inner product space over the real or complex number field  $\mathbb{K}$  ( $\mathbb{K} = \mathbb{R}, \mathbb{C}$ ) and  $x, a \in H$ ,  $r > 0$  are such that

$$\|x - a\| \leq r < \|a\|.$$

Then we have the inequality

$$\begin{aligned} 0 &\leq \|x\| + \|a\| - \|x + a\| \\ &\leq \sqrt{2}r \cdot \sqrt{\frac{\operatorname{Re} \langle x, a \rangle}{\sqrt{\|a\|^2 - r^2} \left( \sqrt{\|a\|^2 - r^2} + \|a\| \right)}}. \end{aligned} \quad (1.63)$$

**Proof.** Using the inequality (1.54), we may write that

$$\|x\| \|a\| \leq \frac{\|a\| \operatorname{Re} \langle x, a \rangle}{\sqrt{\|a\|^2 - r^2}},$$

which gives

$$\begin{aligned}
0 &\leq \|x\| \|a\| - \operatorname{Re} \langle x, a \rangle \\
&\leq \frac{\|a\| - \sqrt{\|a\|^2 - r^2}}{\sqrt{\|a\|^2 - r^2}} \operatorname{Re} \langle x, a \rangle \\
&= \frac{r^2 \operatorname{Re} \langle x, a \rangle}{\sqrt{\|a\|^2 - r^2} \left( \sqrt{\|a\|^2 - r^2} + \|a\| \right)}.
\end{aligned} \tag{1.64}$$

Since

$$(\|x\| + \|a\|)^2 - \|x + a\|^2 = 2(\|x\| \|a\| - \operatorname{Re} \langle x, a \rangle),$$

then by (1.64), we have

$$\begin{aligned}
\|x\| + \|a\| &\leq \sqrt{\|x + a\|^2 + \frac{2r^2 \operatorname{Re} \langle x, a \rangle}{\sqrt{\|a\|^2 - r^2} \left( \sqrt{\|a\|^2 - r^2} + \|a\| \right)}} \\
&\leq \|x + a\| + \sqrt{2}r \cdot \sqrt{\frac{\operatorname{Re} \langle x, a \rangle}{\sqrt{\|a\|^2 - r^2} \left( \sqrt{\|a\|^2 - r^2} + \|a\| \right)}},
\end{aligned}$$

giving the desired inequality (1.63). ■

The following proposition providing a simpler reverse for the triangle inequality also holds [7].

**Proposition 42** *Let  $(H; \langle \cdot, \cdot \rangle)$  be an inner product space over  $\mathbb{K}$  and  $x, y \in H$ ,  $M > m > 0$  such that either*

$$\operatorname{Re} \langle My - x, x - my \rangle \geq 0,$$

*or equivalently,*

$$\left\| x - \frac{M+m}{2} \cdot y \right\| \leq \frac{1}{2} (M-m) \|y\|,$$

*holds. Then we have the inequality*

$$0 \leq \|x\| + \|y\| - \|x + y\| \leq \frac{\sqrt{M} - \sqrt{m}}{\sqrt[4]{mM}} \sqrt{\operatorname{Re} \langle x, y \rangle}. \tag{1.65}$$

**Proof.** Choosing in (1.54),  $a = \frac{M+m}{2}y$ ,  $r = \frac{1}{2}(M-m)\|y\|$  we get

$$\|x\| \|y\| \sqrt{Mm} \leq \frac{M+m}{2} \operatorname{Re} \langle x, y \rangle,$$

giving

$$0 \leq \|x\| \|y\| - \operatorname{Re} \langle x, y \rangle \leq \frac{(\sqrt{M} - \sqrt{m})^2}{2\sqrt{mM}} \operatorname{Re} \langle x, y \rangle.$$

Following the same argument as in the proof of Proposition 41, we deduce the desired inequality (1.65). ■

For some results related to triangle inequality in inner product spaces, see [2], [11], [12] and [13].

#### 1.4.4 Integral Inequalities

Let  $(\Omega, \Sigma, \mu)$  be a measurable space consisting of a set  $\Omega$ ,  $\Sigma$  a  $\sigma$ -algebra of parts and  $\mu$  a countably additive and positive measure on  $\Sigma$  with values in  $\mathbb{R} \cup \{\infty\}$ . Let  $\rho \geq 0$  be a  $g$ -measurable function on  $\Omega$  with  $\int_{\Omega} \rho(s) d\mu(s) = 1$ . Denote by  $L_{\rho}^2(\Omega, \mathbb{K})$  the Hilbert space of all real or complex valued functions defined on  $\Omega$  and  $2 - \rho$ -integrable on  $\Omega$ , i.e.,

$$\int_{\Omega} \rho(s) |f(s)|^2 d\mu(s) < \infty.$$

It is obvious that the following inner product

$$\langle f, g \rangle_{\rho} := \int_{\Omega} \rho(s) f(s) \overline{g(s)} d\mu(s),$$

generates the norm  $\|f\|_{\rho} := \left( \int_{\Omega} \rho(s) |f(s)|^2 d\mu(s) \right)^{\frac{1}{2}}$  of  $L_{\rho}^2(\Omega, \mathbb{K})$ , and all the above results may be stated for integrals.

It is important to observe that, if

$$\operatorname{Re} [f(s) \overline{g(s)}] \geq 0, \quad \text{for } \mu - \text{a.e. } s \in \Omega,$$

then, obviously,

$$\begin{aligned} \operatorname{Re} \langle f, g \rangle_{\rho} &= \operatorname{Re} \left[ \int_{\Omega} \rho(s) f(s) \overline{g(s)} d\mu(s) \right] \\ &= \int_{\Omega} \rho(s) \operatorname{Re} [f(s) \overline{g(s)}] d\mu(s) \geq 0. \end{aligned} \tag{1.66}$$

The reverse is evidently not true in general.

Moreover, if the space is real, i.e.,  $\mathbb{K} = \mathbb{R}$ , then a sufficient condition for (1.66) to hold is:

$$f(s) \geq 0, \quad g(s) \geq 0, \quad \text{for } \mu - \text{a.e. } s \in \Omega.$$

We now provide, by the use of certain results obtained above, some integral inequalities that may be used in practical applications.

**Proposition 43** *Let  $f, g \in L_{\rho}^2(\Omega, \mathbb{K})$  and  $r > 0$  with the properties that*

$$|f(s) - g(s)| \leq r \leq |g(s)|, \quad \text{for } \mu - \text{a.e. } s \in \Omega. \tag{1.67}$$

Then we have the inequalities

$$\begin{aligned}
0 &\leq \int_{\Omega} \rho(s) |f(s)|^2 d\mu(s) \int_{\Omega} \rho(s) |g(s)|^2 d\mu(s) \\
&\quad - \left| \int_{\Omega} \rho(s) f(s) \overline{g(s)} d\mu(s) \right|^2 \\
&\leq \int_{\Omega} \rho(s) |f(s)|^2 d\mu(s) \int_{\Omega} \rho(s) |g(s)|^2 d\mu(s) \\
&\quad - \left[ \int_{\Omega} \rho(s) \operatorname{Re} \left( f(s) \overline{g(s)} \right) d\mu(s) \right]^2 \\
&\leq r^2 \int_{\Omega} \rho(s) |g(s)|^2 d\mu(s).
\end{aligned} \tag{1.68}$$

The constant  $c = 1$  in front of  $r^2$  is best possible.

The proof follows by Theorem 37 and we omit the details [7].

**Proposition 44** Let  $f, g \in L^2_{\rho}(\Omega, \mathbb{K})$  and  $\gamma, \Gamma \in \mathbb{K}$  such that  $\operatorname{Re}(\Gamma\overline{\gamma}) > 0$  and

$$\operatorname{Re} \left[ (\Gamma g(s) - f(s)) \left( \overline{f(s)} - \overline{\gamma g(s)} \right) \right] \geq 0, \quad \text{for } \mu - \text{a.e. } s \in \Omega.$$

Then we have the inequalities

$$\begin{aligned}
&\int_{\Omega} \rho(s) |f(s)|^2 d\mu(s) \int_{\Omega} \rho(s) |g(s)|^2 d\mu(s) \\
&\leq \frac{1}{4} \cdot \frac{\left\{ \operatorname{Re} \left[ (\overline{\Gamma} + \overline{\gamma}) \int_{\Omega} \rho(s) f(s) \overline{g(s)} d\mu(s) \right] \right\}^2}{\operatorname{Re}(\Gamma\overline{\gamma})} \\
&\leq \frac{1}{4} \cdot \frac{|\Gamma + \gamma|^2}{\operatorname{Re}(\Gamma\overline{\gamma})} \left| \int_{\Omega} \rho(s) f(s) \overline{g(s)} d\mu(s) \right|^2.
\end{aligned} \tag{1.69}$$

The constant  $\frac{1}{4}$  is best possible in both inequalities.

The proof follows by Theorem 38 and we omit the details.

**Corollary 45** With the assumptions of Proposition 44, we have the inequality

$$\begin{aligned}
0 &\leq \int_{\Omega} \rho(s) |f(s)|^2 d\mu(s) \int_{\Omega} \rho(s) |g(s)|^2 d\mu(s) \\
&\quad - \left| \int_{\Omega} \rho(s) f(s) \overline{g(s)} d\mu(s) \right|^2 \\
&\leq \frac{1}{4} \cdot \frac{|\Gamma - \gamma|^2}{\operatorname{Re}(\Gamma\overline{\gamma})} \left| \int_{\Omega} \rho(s) f(s) \overline{g(s)} d\mu(s) \right|^2.
\end{aligned} \tag{1.70}$$

The constant  $\frac{1}{4}$  is best possible.

**Remark 46** *If the space is real and we assume, for  $M > m > 0$ , that*

$$mg(s) \leq f(s) \leq Mg(s), \quad \text{for } \mu - \text{a.e. } s \in \Omega,$$

*then, by (1.69) and (1.70), we deduce the inequalities*

$$\begin{aligned} \int_{\Omega} \rho(s) [f(s)]^2 d\mu(s) \int_{\Omega} \rho(s) [g(s)]^2 d\mu(s) \\ \leq \frac{1}{4} \cdot \frac{(M+m)^2}{mM} \left[ \int_{\Omega} \rho(s) f(s) g(s) d\mu(s) \right]^2 \end{aligned} \quad (1.71)$$

*and*

$$\begin{aligned} 0 &\leq \int_{\Omega} \rho(s) [f(s)]^2 d\mu(s) \int_{\Omega} \rho(s) [g(s)]^2 d\mu(s) \\ &\quad - \left[ \int_{\Omega} \rho(s) f(s) g(s) d\mu(s) \right]^2 \\ &\leq \frac{1}{4} \cdot \frac{(M-m)^2}{mM} \left[ \int_{\Omega} \rho(s) f(s) g(s) d\mu(s) \right]^2. \end{aligned} \quad (1.72)$$

*The inequality (1.71) is known in the literature as Cassel's inequality.*

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## 1.5 More Reverses of Schwarz's Inequality

### 1.5.1 General Results

The following result holds [5].

**Theorem 47** *Let  $(H; \langle \cdot, \cdot \rangle)$  be an inner product space over the real or complex number field  $\mathbb{K}$ ,  $x, a \in H$  and  $r > 0$ . If*

$$x \in \bar{B}(a, r) := \{z \in H \mid \|z - a\| \leq r\}, \quad (1.73)$$

*then we have the inequalities:*

$$\begin{aligned} 0 &\leq \|x\| \|a\| - |\langle x, a \rangle| \leq \|x\| \|a\| - |\operatorname{Re} \langle x, a \rangle| \\ &\leq \|x\| \|a\| - \operatorname{Re} \langle x, a \rangle \leq \frac{1}{2} r^2. \end{aligned} \quad (1.74)$$

*The constant  $\frac{1}{2}$  is best possible in (1.74) in the sense that it cannot be replaced by a smaller constant.*

**Proof.** The condition (1.73) is clearly equivalent to

$$\|x\|^2 + \|a\|^2 \leq 2 \operatorname{Re} \langle x, a \rangle + r^2. \quad (1.75)$$

Using the elementary inequality

$$2 \|x\| \|a\| \leq \|x\|^2 + \|a\|^2, \quad a, x \in H$$

and (1.75), we deduce

$$2 \|x\| \|a\| \leq 2 \operatorname{Re} \langle x, a \rangle + r^2,$$

giving the last inequality in (1.74). The other inequalities are obvious.

To prove the sharpness of the constant  $\frac{1}{2}$ , assume that

$$0 \leq \|x\| \|a\| - \operatorname{Re} \langle x, a \rangle \leq cr^2 \quad (1.76)$$

for any  $x, a \in H$  and  $r > 0$  satisfying (1.73).

Assume that  $a, e \in H$ ,  $\|a\| = \|e\| = 1$  and  $e \perp a$ . If  $r = \sqrt{\varepsilon}$ ,  $\varepsilon > 0$  and if we define  $x = a + \sqrt{\varepsilon}e$ , then  $\|x - a\| = \sqrt{\varepsilon} = r$  showing that the condition (1.73) is fulfilled.

On the other hand,

$$\begin{aligned} \|x\| \|a\| - \operatorname{Re} \langle x, a \rangle &= \sqrt{\|a + \sqrt{\varepsilon}e\|^2} - \operatorname{Re} \langle a + \sqrt{\varepsilon}e, a \rangle \\ &= \sqrt{\|a\|^2 + \varepsilon \|e\|^2} - \|a\|^2 \\ &= \sqrt{1 + \varepsilon} - 1. \end{aligned}$$



Utilising (1.76), we conclude that

$$\sqrt{1+\varepsilon} - 1 \leq c\varepsilon \quad \text{for any } \varepsilon > 0. \quad (1.77)$$

Multiplying (1.77) by  $\sqrt{1+\varepsilon} + 1 > 0$  and then dividing by  $\varepsilon > 0$ , we get

$$(\sqrt{1+\varepsilon} + 1) c \geq 1 \quad \text{for any } \varepsilon > 0. \quad (1.78)$$

Letting  $\varepsilon \rightarrow 0+$  in (1.78), we deduce  $c \geq \frac{1}{2}$ , and the theorem is proved. ■

The following result also holds [5].

**Theorem 48** *Let  $(H; \langle \cdot, \cdot \rangle)$  be an inner product space over  $\mathbb{K}$  and  $x, y \in H$ ,  $\gamma, \Gamma \in \mathbb{K}$  ( $\Gamma \neq -\gamma$ ) so that either*

$$\operatorname{Re} \langle \Gamma y - x, x - \gamma y \rangle \geq 0, \quad (1.79)$$

*or equivalently,*

$$\left\| x - \frac{\gamma + \Gamma}{2} y \right\| \leq \frac{1}{2} |\Gamma - \gamma| \|y\|, \quad (1.80)$$

*holds. Then we have the inequalities*

$$\begin{aligned} 0 &\leq \|x\| \|y\| - |\langle x, y \rangle| \\ &\leq \|x\| \|y\| - \left| \operatorname{Re} \left[ \frac{\bar{\Gamma} + \bar{\gamma}}{|\Gamma + \gamma|} \langle x, y \rangle \right] \right| \\ &\leq \|x\| \|y\| - \operatorname{Re} \left[ \frac{\bar{\Gamma} + \bar{\gamma}}{|\Gamma + \gamma|} \langle x, y \rangle \right] \\ &\leq \frac{1}{4} \cdot \frac{|\Gamma - \gamma|^2}{|\Gamma + \gamma|} \|y\|^2. \end{aligned} \quad (1.81)$$

*The constant  $\frac{1}{4}$  in the last inequality is best possible.*

**Proof.** Consider for  $a, y \neq 0$ ,  $a = \frac{\Gamma + \gamma}{2} \cdot y$  and  $r = \frac{1}{2} |\Gamma - \gamma| \|y\|$ . Thus from (1.74), we get

$$\begin{aligned} 0 &\leq \|x\| \left| \frac{\Gamma + \gamma}{2} \|y\| - \left| \frac{\Gamma + \gamma}{2} \right| |\langle x, y \rangle| \right| \\ &\leq \|x\| \left| \frac{\Gamma + \gamma}{2} \|y\| - \left| \operatorname{Re} \left[ \frac{\bar{\Gamma} + \bar{\gamma}}{|\Gamma + \gamma|} \langle x, y \rangle \right] \right| \right| \\ &\leq \|x\| \left| \frac{\Gamma + \gamma}{2} \|y\| - \operatorname{Re} \left[ \frac{\bar{\Gamma} + \bar{\gamma}}{|\Gamma + \gamma|} \langle x, y \rangle \right] \right| \\ &\leq \frac{1}{8} \cdot |\Gamma - \gamma|^2 \|y\|^2. \end{aligned}$$

Dividing by  $\frac{1}{2} |\Gamma + \gamma| \geq 0$ , we deduce the desired inequality (1.81).

To prove the sharpness of the constant  $\frac{1}{4}$ , assume that there exists a  $c > 0$  such that:

$$\|x\| \|y\| - \operatorname{Re} \left[ \frac{\bar{\Gamma} + \bar{\gamma}}{|\Gamma + \gamma|} \langle x, y \rangle \right] \leq c \cdot \frac{|\Gamma - \gamma|^2}{|\Gamma + \gamma|} \|y\|^2, \quad (1.82)$$

provided either (1.79) or (1.80) holds.

Consider the real inner product space  $(\mathbb{R}^2, \langle \cdot, \cdot \rangle)$  with  $\langle \bar{\mathbf{x}}, \bar{\mathbf{y}} \rangle = x_1 y_1 + x_2 y_2$ ,  $\bar{\mathbf{x}} = (x_1, x_2)$ ,  $\bar{\mathbf{y}} = (y_1, y_2) \in \mathbb{R}^2$ . Let  $\bar{\mathbf{y}} = (1, 1)$  and  $\Gamma, \gamma > 0$  with  $\Gamma > \gamma$ . Then, by (1.82), we deduce

$$\sqrt{2} \sqrt{x_1^2 + x_2^2} - (x_1 + x_2) \leq 2c \cdot \frac{(\Gamma - \gamma)^2}{\Gamma + \gamma}. \quad (1.83)$$

If  $x_1 = \Gamma$ ,  $x_2 = \gamma$ , then

$$\langle \Gamma \bar{\mathbf{y}} - \bar{\mathbf{x}}, \bar{\mathbf{x}} - \gamma \bar{\mathbf{y}} \rangle = (\Gamma - x_1)(x_1 - \gamma) + (\Gamma - x_2)(x_2 - \gamma) = 0,$$

showing that the condition (1.79) is valid. Replacing  $x_1$  and  $x_2$  in (1.83), we deduce

$$\sqrt{2} \sqrt{\Gamma^2 + \gamma^2} - (\Gamma + \gamma) \leq 2c \frac{(\Gamma - \gamma)^2}{\Gamma + \gamma}. \quad (1.84)$$

If in (1.84) we choose  $\Gamma = 1 + \varepsilon$ ,  $\gamma = 1 - \varepsilon$  with  $\varepsilon \in (0, 1)$ , then we have

$$2\sqrt{1 + \varepsilon^2} - 2 \leq 2c \frac{4\varepsilon^2}{2},$$

giving

$$\sqrt{1 + \varepsilon^2} - 1 \leq 2c\varepsilon^2. \quad (1.85)$$

Finally, multiplying (1.85) with  $\sqrt{1 + \varepsilon^2} + 1 > 0$  and thus dividing by  $\varepsilon^2$ , we deduce

$$1 \leq 2c \left( \sqrt{1 + \varepsilon^2} + 1 \right) \quad \text{for any } \varepsilon \in (0, 1). \quad (1.86)$$

Letting  $\varepsilon \rightarrow 0+$  in (1.86) we get  $c \geq \frac{1}{4}$ , and the sharpness of the constant is proved. ■

### 1.5.2 Reverses of the Triangle Inequality

The following reverse of the triangle inequality in inner product spaces holds [5].

**Proposition 49** *Let  $(H; \langle \cdot, \cdot \rangle)$  be an inner product space over the real or complex number field  $\mathbb{K}$ ,  $x, a \in H$  and  $r > 0$ . If  $\|x - a\| \leq r$ , then we have the inequality*

$$0 \leq \|x\| + \|a\| - \|x + a\| \leq r. \quad (1.87)$$

**Proof.** Since

$$(\|x\| + \|a\|)^2 - \|x + a\|^2 \leq 2(\|x\| \|a\| - \operatorname{Re} \langle x, a \rangle),$$

then by Theorem 47 we deduce

$$(\|x\| + \|a\|)^2 - \|x + a\|^2 \leq r^2,$$

from where we obtain

$$\|x\| + \|a\| \leq \sqrt{r^2 + \|x + a\|^2} \leq r + \|x + a\|,$$

giving the desired result (1.87). ■

We may state the following result [5].

**Proposition 50** *Let  $(H; \langle \cdot, \cdot \rangle)$  be an inner product space over  $\mathbb{K}$  and  $x, y \in H$ ,  $M > m > 0$  such that either*

$$\operatorname{Re} \langle My - x, x - my \rangle \geq 0,$$

*or equivalently,*

$$\left\| x - \frac{M+m}{2}y \right\| \leq \frac{1}{2}(M-m)\|y\|,$$

*holds. Then we have the inequality*

$$0 \leq \|x\| + \|y\| - \|x + y\| \leq \frac{\sqrt{2}}{2} \cdot \frac{(M-m)}{\sqrt{M+m}} \|y\|. \quad (1.88)$$

**Proof.** By Theorem 48 for  $\Gamma = M$ ,  $\gamma = m$ , we have the inequality

$$\|x\| \|y\| - \operatorname{Re} \langle x, y \rangle \leq \frac{1}{4} \cdot \frac{(M-m)^2}{(M+m)} \|y\|^2.$$

Then we may state that

$$\begin{aligned} (\|x\| + \|y\|)^2 - \|x + y\|^2 &= 2(\|x\| \|y\| - \operatorname{Re} \langle x, y \rangle) \\ &\leq \frac{1}{2} \cdot \frac{(M-m)^2}{M+m} \|y\|^2, \end{aligned}$$

from where we get

$$\begin{aligned} \|x\| + \|y\| &\leq \sqrt{\frac{1}{2} \cdot \frac{(M-m)^2}{M+m} \|y\|^2 + \|x + y\|^2} \\ &\leq \|x + y\| + \frac{(M-m)}{\sqrt{2(M+m)}} \|y\|, \end{aligned}$$

giving the desired inequality (1.88). ■

### 1.5.3 Integral Inequalities

We provide now, by the use of certain results obtained above, some integral inequalities that may be used in practical applications.

**Proposition 51** *Let  $f, g \in L^2_\rho(\Omega, \mathbb{K})$  and  $r > 0$  with the property that*

$$|f(s) - g(s)| \leq r \quad \text{for, } \mu - \text{a.e. } s \in \Omega.$$

*Then we have the inequalities*

$$\begin{aligned} 0 &\leq \left[ \int_{\Omega} \rho(s) |f(s)|^2 d\mu(s) \int_{\Omega} \rho(s) |g(s)|^2 d\mu(s) \right]^{\frac{1}{2}} \\ &\quad - \left| \int_{\Omega} \rho(s) f(s) \overline{g(s)} d\mu(s) \right| \\ &\leq \left[ \int_{\Omega} \rho(s) |f(s)|^2 d\mu(s) \int_{\Omega} \rho(s) |g(s)|^2 d\mu(s) \right]^{\frac{1}{2}} \\ &\quad - \left| \int_{\Omega} \rho(s) \operatorname{Re} [f(s) \overline{g(s)}] d\mu(s) \right| \\ &\leq \left[ \int_{\Omega} \rho(s) |f(s)|^2 d\mu(s) \int_{\Omega} \rho(s) |g(s)|^2 d\mu(s) \right]^{\frac{1}{2}} \\ &\quad - \int_{\Omega} \rho(s) \operatorname{Re} [f(s) \overline{g(s)}] d\mu(s) \\ &\leq \frac{1}{2} r^2. \end{aligned} \tag{1.89}$$

*The constant  $\frac{1}{2}$  is best possible in (1.89).*

The proof follows by Theorem 47, and we omit the details.

**Proposition 52** *Let  $f, g \in L^2_\rho(\Omega, \mathbb{K})$  and  $\gamma, \Gamma \in \mathbb{K}$  so that  $\Gamma \neq -\gamma$ , and*

$$\operatorname{Re} \left[ (\Gamma g(s) - f(s)) (\overline{f(s)} - \overline{\gamma g(s)}) \right] \geq 0, \quad \text{for } \mu - \text{a.e. } s \in \Omega.$$

*Then we have the inequalities*

$$\begin{aligned} 0 &\leq \left[ \int_{\Omega} \rho(s) |f(s)|^2 d\mu(s) \int_{\Omega} \rho(s) |g(s)|^2 d\mu(s) \right]^{\frac{1}{2}} \\ &\quad - \left| \int_{\Omega} \rho(s) f(s) \overline{g(s)} d\mu(s) \right| \\ &\leq \left[ \int_{\Omega} \rho(s) |f(s)|^2 d\mu(s) \int_{\Omega} \rho(s) |g(s)|^2 d\mu(s) \right]^{\frac{1}{2}} \\ &\quad - \left| \operatorname{Re} \left[ \frac{\bar{\Gamma} + \bar{\gamma}}{|\Gamma + \gamma|} \int_{\Omega} \rho(s) f(s) \overline{g(s)} d\mu(s) \right] \right| \end{aligned} \tag{1.90}$$

$$\begin{aligned}
&\leq \left[ \int_{\Omega} \rho(s) |f(s)|^2 d\mu(s) \int_{\Omega} \rho(s) |g(s)|^2 d\mu(s) \right]^{\frac{1}{2}} \\
&\quad - \operatorname{Re} \left[ \frac{\bar{\Gamma} + \bar{\gamma}}{|\Gamma + \gamma|} \int_{\Omega} \rho(s) f(s) \overline{g(s)} d\mu(s) \right] \\
&\leq \frac{1}{4} \cdot \frac{|\Gamma - \gamma|^2}{|\Gamma + \gamma|} \int_{\Omega} \rho(s) |g(s)|^2 d\mu(s).
\end{aligned}$$

The constant  $\frac{1}{4}$  is best possible.

**Remark 53** If the space is real and we assume, for  $M > m > 0$ , that

$$mg(s) \leq f(s) \leq Mg(s), \quad \text{for } \mu - \text{a.e. } s \in \Omega, \quad (1.91)$$

then, by (1.90), we deduce the inequality:

$$\begin{aligned}
0 &\leq \left[ \int_{\Omega} \rho(s) |f(s)|^2 d\mu(s) \int_{\Omega} \rho(s) |g(s)|^2 d\mu(s) \right]^{\frac{1}{2}} \\
&\quad - \left| \int_{\Omega} \rho(s) f(s) \overline{g(s)} d\mu(s) \right| \\
&\leq \frac{1}{4} \cdot \frac{(M - m)^2}{M + m} \int_{\Omega} \rho(s) |g(s)|^2 d\mu(s).
\end{aligned}$$

The constant  $\frac{1}{4}$  is best possible.

The following reverse of the triangle inequality for integrals holds.

**Proposition 54** Assume that the functions  $f, g \in L^2_{\rho}(\Omega, \mathbb{K})$  satisfy (1.91). Then we have the inequality

$$\begin{aligned}
0 &\leq \left( \int_{\Omega} \rho(s) |f(s)|^2 d\mu(s) \right)^{\frac{1}{2}} + \left( \int_{\Omega} \rho(s) |g(s)|^2 d\mu(s) \right)^{\frac{1}{2}} \\
&\quad - \left( \int_{\Omega} \rho(s) |f(s) + g(s)|^2 d\mu(s) \right)^{\frac{1}{2}} \\
&\leq \frac{\sqrt{2}}{2} \cdot \frac{M - m}{\sqrt{M + m}} \left( \int_{\Omega} \rho(s) |g(s)|^2 d\mu(s) \right)^{\frac{1}{2}}.
\end{aligned}$$

The proof follows by Proposition 50.

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## Chapter 2

# Inequalities of the Grüss Type

### 2.1 Introduction

Over the last five years, the development of Grüss type inequalities has experienced a surge, having been stimulated by their applications in different branches of Applied Mathematics including: in perturbed quadrature rules (see for example [5], [3]) and in the approximation of integral transforms (see [14], [17]) and the references therein.

For two Lebesgue integrable functions  $f, g : [a, b] \rightarrow \mathbb{R}$ , consider the Čebyšev functional:

$$T(f, g) := \frac{1}{b-a} \int_a^b f(t) g(t) dt - \frac{1}{b-a} \int_a^b f(t) dt \cdot \frac{1}{b-a} \int_a^b g(t) dt.$$

In 1934, G. Grüss [16] showed that

$$|T(f, g)| \leq \frac{1}{4} (M - m) (N - n), \quad (2.1)$$

provided  $m, M, n, N$  are real numbers with the property

$$-\infty < m \leq f \leq M < \infty, \quad -\infty < n \leq g \leq N < \infty \quad \text{a.e. on } [a, b]. \quad (2.2)$$

The constant  $\frac{1}{4}$  is best possible in (2.1) in the sense that it cannot be replaced by a smaller one. Another less well known inequality for  $T(f, g)$  was derived in 1882 by Čebyšev [4] under the assumption that  $f', g'$  exist and are continuous in  $[a, b]$  and is given by

$$|T(f, g)| \leq \frac{1}{12} \|f'\|_\infty \|g'\|_\infty (b-a)^2, \quad (2.3)$$

where  $\|f'\|_\infty := \sup_{t \in [a, b]} |f'(t)|$ .

The constant  $\frac{1}{12}$  cannot be improved in the general case.

Čebyšev's inequality (2.3) also holds if  $f, g : [a, b] \rightarrow \mathbb{R}$  are assumed to be absolutely continuous and  $f', g' \in L_\infty [a, b]$ .

In 1970, A.M. Ostrowski [19] proved, amongst others, the following result that is in a sense a combination of the Čebyšev and Grüss results

$$|T(f, g)| \leq \frac{1}{8} (b - a) (M - m) \|g'\|_\infty,$$

provided  $f$  is Lebesgue integrable on  $[a, b]$  and satisfying (2.2) with  $g : [a, b] \rightarrow \mathbb{R}$  being absolutely continuous and  $g' \in L_\infty [a, b]$ . Here the constant  $\frac{1}{8}$  is also sharp.

Finally, let us recall a result by Lupaş (see for example [20, p. 210]), which states that:

$$|T(f, g)| \leq \frac{1}{\pi^2} \|f'\|_2 \|g'\|_2 (b - a),$$

provided  $f, g$  are absolutely continuous and  $f', g' \in L_2 [a, b]$ . The constant  $\frac{1}{\pi^2}$  is the best possible here also.

For other Grüss type integral inequalities, see the books [18], [20], and the papers [6] – [13], where further references are given.

In [1], P. Cerone has obtained the following identity that involves a Stieltjes integral (Lemma 2.1, p. 3):

**Lemma 55** *Let  $f, g : [a, b] \rightarrow \mathbb{R}$ , where  $f$  is of bounded variation and  $g$  is continuous on  $[a, b]$ , then the  $T(f, g)$  satisfies the identity,*

$$T(f, g) = \frac{1}{(b - a)^2} \int_a^b \Psi(t) df(t), \quad (2.4)$$

where

$$\Psi(t) := (t - a) A(t, b) - (b - t) A(a, t),$$

with

$$A(c, d) := \int_c^d g(x) dx.$$

Using this representation and the properties of Stieltjes integrals he obtained the following result in bounding the functional  $T(\cdot, \cdot)$  (Theorem 2.5, p. 4):

**Theorem 56** *With the assumptions in Lemma 55, we have:*

$$|T(f, g)| \leq \frac{1}{(b - a)^2} \cdot \begin{cases} \sup_{t \in [a, b]} |\Psi(t)| \bigvee_a^b(f), \\ L \int_a^b |\Psi(t)| dt, & \text{for } L - \text{Lipschitzian}; \\ \int_a^b |\Psi(t)| df(t), & \text{for } f \text{ monotonic nondecreasing,} \end{cases}$$

where  $\bigvee_a^b(f)$  denotes the total variation of  $f$  on  $[a, b]$ .



Now, if we use the function  $\varphi : (a, b) \rightarrow \mathbb{R}$ ,

$$\varphi(t) := D(g; a, t, b) = \frac{\int_t^b g(x) dx}{b-t} - \frac{\int_a^t g(x) dx}{t-a}, \quad (2.5)$$

then by (2.4) we may obtain the identity:

$$T(f, g) = \frac{1}{(b-a)^2} \int_a^b (t-a)(b-t) \varphi(t) df(t).$$

In [3] various upper bounds for  $|T(f, g)|$  have been given, from which we would like to mention only the following ones

**Theorem 57** *Let  $f : [a, b] \rightarrow \mathbb{R}$  be a function of bounded variation and  $g : [a, b] \rightarrow \mathbb{R}$  an absolutely continuous function so that  $\varphi$  is bounded on  $(a, b)$ . Then one has the inequality:*

$$|T(f, g)| \leq \frac{1}{4} \|\varphi\|_\infty \bigvee_a^b(f),$$

where  $\varphi$  is as given by (2.5) and

$$\|\varphi\|_\infty := \sup_{t \in (a, b)} |\varphi(t)|.$$

The case of Lipschitzian functions  $f : [a, b] \rightarrow \mathbb{R}$  is embodied in the following theorem as well [3].

**Theorem 58** *Let  $f : [a, b] \rightarrow \mathbb{R}$  be an  $L$ -Lipschitzian function on  $[a, b]$  and  $g : [a, b] \rightarrow \mathbb{R}$  an absolutely continuous function on  $[a, b]$ . Then*

$$|T(f, g)| \leq \begin{cases} L \frac{(b-a)^3}{6} \|\varphi\|_\infty & \text{if } \varphi \in L_\infty[a, b]; \\ L(b-a)^{\frac{1}{q}} [B(q+1, q+1)]^{\frac{1}{q}} \|\varphi\|_p, & p > 1, \frac{1}{p} + \frac{1}{q} = 1 \\ & \text{if } \varphi \in L_p[a, b]; \\ \frac{L}{4} \|\varphi\|_1, & \text{if } \varphi \in L_1[a, b], \end{cases}$$

where  $\|\cdot\|_p$  are the usual Lebesgue  $p$ -norms on  $[a, b]$  and  $B(\cdot, \cdot)$  is Euler's Beta function.

Finally, the following result containing Stieltjes integral holds [3]:

**Theorem 59** Let  $f : [a, b] \rightarrow \mathbb{R}$  be a monotonic nondecreasing function on  $[a, b]$ . If  $g$  is continuous, then one has the inequality:

$$|T(f, g)| \leq \begin{cases} \frac{1}{4} \int_a^b |\varphi(t)| df(t) \\ \frac{1}{(b-a)^2} \left( \int_a^b [(b-t)(t-a)]^q df(t) \right)^{\frac{1}{q}} \left( \int_a^b |\varphi(t)|^p df(t) \right)^{\frac{1}{p}}, \\ \quad p > 1, \frac{1}{p} + \frac{1}{q} = 1; \\ \frac{1}{(b-a)^2} \sup_{t \in [a, b]} |\varphi(t)| \int_a^b (t-a)(b-t) df(t). \end{cases}$$

In [12], the authors have considered the following functional

$$D(f; u) := \int_a^b f(x) du(x) - [u(b) - u(a)] \cdot \frac{1}{b-a} \int_a^b f(t) dt,$$

provided that the involved integrals exist.

In the same paper, the following result in estimating the above functional has been obtained.

**Theorem 60** Let  $f, u : [a, b] \rightarrow \mathbb{R}$  be such that  $u$  is Lipschitzian on  $[a, b]$ , i.e.,

$$|u(x) - u(y)| \leq L|x - y| \quad \text{for any } x, y \in [a, b] \quad (L > 0)$$

and  $f$  is Riemann integrable on  $[a, b]$ . If  $m, M \in \mathbb{R}$  are such that

$$m \leq f(x) \leq M \quad \text{for any } x, y \in [a, b],$$

then we have the inequality

$$|D(f; u)| \leq \frac{1}{2} L (M - m) (b - a).$$

The constant  $\frac{1}{2}$  is sharp in the sense that it cannot be replaced by a smaller constant.

In [13], the following result complementing the above one was obtained.

**Theorem 61** Let  $f, u : [a, b] \rightarrow \mathbb{R}$  be such that  $u : [a, b] \rightarrow \mathbb{R}$  is of bounded variation in  $[a, b]$  and  $f : [a, b] \rightarrow \mathbb{R}$  is  $K$ -Lipschitzian ( $K > 0$ ). Then we have the inequality

$$|D(f; u)| \leq \frac{1}{2} K (b - a) \bigvee_a^b(u).$$

The constant  $\frac{1}{2}$  is sharp in the above sense.

The main aim of this section is to survey some recent inequalities of the Grüss type holding in the general setting of inner product spaces. Natural applications for Lebesgue integrals in measure spaces are presented as well.

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## 2.2 Grüss' Inequality in Inner Product Spaces

### 2.2.1 Introduction

In [1], the author has proved the following Grüss' type inequality in real or complex inner product spaces.

**Theorem 62** *Let  $(H, \langle \cdot, \cdot \rangle)$  be an inner product space over  $\mathbb{K}$  ( $\mathbb{K} = \mathbb{R}, \mathbb{C}$ ) and  $e \in H, \|e\| = 1$ . If  $\varphi, \gamma, \Phi, \Gamma$  are real or complex numbers and  $x, y$  are vectors in  $H$  such that the conditions*

$$\operatorname{Re} \langle \Phi e - x, x - \varphi e \rangle \geq 0 \text{ and } \operatorname{Re} \langle \Gamma e - y, y - \gamma e \rangle \geq 0 \quad (2.6)$$

*hold, then we have the inequality*

$$|\langle x, y \rangle - \langle x, e \rangle \langle e, y \rangle| \leq \frac{1}{4} |\Phi - \varphi| \cdot |\Gamma - \gamma|. \quad (2.7)$$

*The constant  $\frac{1}{4}$  is best possible in the sense that it cannot be replaced by a smaller constant.*

Some particular cases of interest for integrable functions with real or complex values and the corresponding discrete versions are listed below.

**Corollary 63** *Let  $f, g : [a, b] \rightarrow \mathbb{K}$  ( $\mathbb{K} = \mathbb{R}, \mathbb{C}$ ) be Lebesgue integrable and such that*

$$\operatorname{Re} \left[ (\Phi - f(x)) (\overline{f(x)} - \overline{\varphi}) \right] \geq 0, \quad \operatorname{Re} \left[ (\Gamma - g(x)) (\overline{g(x)} - \overline{\gamma}) \right] \geq 0$$

*for a.e.  $x \in [a, b]$ , where  $\varphi, \gamma, \Phi, \Gamma$  are real or complex numbers and  $\bar{z}$  denotes the complex conjugate of  $z$ . Then we have the inequality*

$$\left| \frac{1}{b-a} \int_a^b f(x) \overline{g(x)} dx - \frac{1}{b-a} \int_a^b f(x) dx \cdot \frac{1}{b-a} \int_a^b \overline{g(x)} dx \right| \leq \frac{1}{4} |\Phi - \varphi| \cdot |\Gamma - \gamma|.$$

*The constant  $\frac{1}{4}$  is best possible.*

The discrete case is embodied in

**Corollary 64** *Let  $\mathbf{x}, \mathbf{y} \in \mathbb{K}^n$  and  $\varphi, \gamma, \Phi, \Gamma$  are real or complex numbers such that*

$$\operatorname{Re} [(\Phi - x_i) (\overline{x_i} - \overline{\varphi})] \geq 0, \quad \operatorname{Re} [(\Gamma - y_i) (\overline{y_i} - \overline{\gamma})] \geq 0$$

*for each  $i \in \{1, \dots, n\}$ . Then we have the inequality*

$$\left| \frac{1}{n} \sum_{i=1}^n x_i \overline{y_i} - \frac{1}{n} \sum_{i=1}^n x_i \cdot \frac{1}{n} \sum_{i=1}^n \overline{y_i} \right| \leq \frac{1}{4} |\Phi - \varphi| \cdot |\Gamma - \gamma|.$$

*The constant  $\frac{1}{4}$  is best possible.*

For other applications of Theorem 62, see the recent paper [2].

In the present section, by following [3], we show that the condition (2.6) may be replaced by an equivalent but simpler assumption and a new proof of Theorem 62 is produced. A refinement of the Grüss type inequality (2.7), some companions and applications for integrals are pointed out as well.

### 2.2.2 An Equivalent Assumption

The following lemma holds [3].

**Lemma 65** *Let  $a, x, A$  be vectors in the inner product space  $(H, \langle \cdot, \cdot \rangle)$  over  $\mathbb{K}$  ( $\mathbb{K} = \mathbb{R}, \mathbb{C}$ ) with  $a \neq A$ . Then*

$$\operatorname{Re} \langle A - x, x - a \rangle \geq 0$$

*if and only if*

$$\left\| x - \frac{a + A}{2} \right\| \leq \frac{1}{2} \|A - a\|.$$

**Proof.** Define

$$I_1 := \operatorname{Re} \langle A - x, x - a \rangle, \quad I_2 := \frac{1}{4} \|A - a\|^2 - \left\| x - \frac{a + A}{2} \right\|^2.$$

A simple calculation shows that

$$I_1 = I_2 = \operatorname{Re} [\langle x, a \rangle + \langle A, x \rangle] - \operatorname{Re} \langle A, a \rangle - \|x\|^2$$

and thus, obviously,  $I_1 \geq 0$  iff  $I_2 \geq 0$  showing the required equivalence. ■

The following corollary is obvious

**Corollary 66** *Let  $x, e \in H$  with  $\|e\| = 1$  and  $\delta, \Delta \in \mathbb{K}$  with  $\delta \neq \Delta$ . Then*

$$\operatorname{Re} \langle \Delta e - x, x - \delta e \rangle \geq 0$$

*iff*

$$\left\| x - \frac{\delta + \Delta}{2} \cdot e \right\| \leq \frac{1}{2} |\Delta - \delta|.$$

**Remark 67** *If  $H = \mathbb{C}$ , then*

$$\operatorname{Re} [(A - x)(\bar{x} - \bar{a})] \geq 0$$

*if and only if*

$$\left| x - \frac{a + A}{2} \right| \leq \frac{1}{2} |A - a|,$$

*where  $a, x, A \in \mathbb{C}$ . If  $H = \mathbb{R}$ , and  $A > a$  then  $a \leq x \leq A$  if and only if  $\left| x - \frac{a+A}{2} \right| \leq \frac{1}{2} |A - a|$ .*

The following lemma also holds [3].

**Lemma 68** *Let  $x, e \in H$  with  $\|e\| = 1$ . Then one has the following representation*

$$0 \leq \|x\|^2 - |\langle x, e \rangle|^2 = \inf_{\lambda \in \mathbb{K}} \|x - \lambda e\|^2. \quad (2.8)$$

**Proof.** Observe, for any  $\lambda \in \mathbb{K}$ , that

$$\begin{aligned} \langle x - \lambda e, x - \langle x, e \rangle e \rangle &= \|x\|^2 - |\langle x, e \rangle|^2 - \lambda \left[ \langle e, x \rangle - \langle e, x \rangle \|e\|^2 \right] \\ &= \|x\|^2 - |\langle x, e \rangle|^2. \end{aligned}$$

Using Schwarz's inequality, we have

$$\begin{aligned} \left[ \|x\|^2 - |\langle x, e \rangle|^2 \right]^2 &= |\langle x - \lambda e, x - \langle x, e \rangle e \rangle|^2 \\ &\leq \|x - \lambda e\|^2 \|x - \langle x, e \rangle e\|^2 \\ &= \|x - \lambda e\|^2 \left[ \|x\|^2 - |\langle x, e \rangle|^2 \right], \end{aligned}$$

giving the bound

$$\|x\|^2 - |\langle x, e \rangle|^2 \leq \|x - \lambda e\|^2, \quad \lambda \in \mathbb{K}. \quad (2.9)$$

Taking the infimum in (2.9) over  $\lambda \in \mathbb{K}$ , we deduce

$$\|x\|^2 - |\langle x, e \rangle|^2 \leq \inf_{\lambda \in \mathbb{K}} \|x - \lambda e\|^2.$$

Since, for  $\lambda_0 = \langle x, e \rangle$ , we get  $\|x - \lambda_0 e\|^2 = \|x\|^2 - |\langle x, e \rangle|^2$ , then the representation (2.8) is proved. ■

We are now able to provide a different proof for the Grüss type inequality in inner product spaces (mentioned in the Introduction), than the one from the paper [1].

**Theorem 69** *Let  $(H, \langle \cdot, \cdot \rangle)$  be an inner product space over  $\mathbb{K}$  ( $\mathbb{K} = \mathbb{R}, \mathbb{C}$ ) and  $e \in H, \|e\| = 1$ . If  $\varphi, \gamma, \Phi, \Gamma$  are real or complex numbers and  $x, y$  are vectors in  $H$  such that the conditions (2.6) hold, or equivalently, the following assumptions*

$$\left\| x - \frac{\varphi + \Phi}{2} \cdot e \right\| \leq \frac{1}{2} |\Phi - \varphi|, \quad \left\| y - \frac{\gamma + \Gamma}{2} \cdot e \right\| \leq \frac{1}{2} |\Gamma - \gamma| \quad (2.10)$$

*are valid, then one has the inequality*

$$|\langle x, y \rangle - \langle x, e \rangle \langle e, y \rangle| \leq \frac{1}{4} |\Phi - \varphi| \cdot |\Gamma - \gamma|. \quad (2.11)$$

*The constant  $\frac{1}{4}$  is best possible.*

**Proof.** It can be easily shown (see for example the proof of Theorem 1 from [1]) that

$$|\langle x, y \rangle - \langle x, e \rangle \langle e, y \rangle| \leq \left[ \|x\|^2 - |\langle x, e \rangle|^2 \right]^{\frac{1}{2}} \left[ \|y\|^2 - |\langle y, e \rangle|^2 \right]^{\frac{1}{2}}, \quad (2.12)$$

for any  $x, y \in H$  and  $e \in H, \|e\| = 1$ . Using Lemma 68 and the conditions (2.10) we obviously have that

$$\left[ \|x\|^2 - |\langle x, e \rangle|^2 \right]^{\frac{1}{2}} = \inf_{\lambda \in \mathbb{K}} \|x - \lambda e\| \leq \left\| x - \frac{\varphi + \Phi}{2} \cdot e \right\| \leq \frac{1}{2} |\Phi - \varphi|$$

and

$$\left[ \|y\|^2 - |\langle y, e \rangle|^2 \right]^{\frac{1}{2}} = \inf_{\lambda \in \mathbb{K}} \|y - \lambda e\| \leq \left\| y - \frac{\gamma + \Gamma}{2} \cdot e \right\| \leq \frac{1}{2} |\Gamma - \gamma|$$

and by (2.12) the desired inequality (2.11) is obtained.

The fact that  $\frac{1}{4}$  is the best possible constant, has been shown in [1] and we omit the details. ■

### 2.2.3 A Refinement of the Grüss Inequality

The following result improving (2.6) holds [3].

**Theorem 70** *Let  $(H, \langle \cdot, \cdot \rangle)$  be an inner product space over  $\mathbb{K}$  ( $\mathbb{K} = \mathbb{R}, \mathbb{C}$ ) and  $e \in H, \|e\| = 1$ . If  $\varphi, \gamma, \Phi, \Gamma$  are real or complex numbers and  $x, y$  are vectors in  $H$  such that the conditions (2.6), or equivalently, (2.10) hold, then we have the inequality*

$$\begin{aligned} & |\langle x, y \rangle - \langle x, e \rangle \langle e, y \rangle| \\ & \leq \frac{1}{4} |\Phi - \varphi| \cdot |\Gamma - \gamma| - [\operatorname{Re} \langle \Phi e - x, x - \varphi e \rangle]^{\frac{1}{2}} [\operatorname{Re} \langle \Gamma e - y, y - \gamma e \rangle]^{\frac{1}{2}} \\ & \leq \left( \frac{1}{4} |\Phi - \varphi| \cdot |\Gamma - \gamma| \right). \end{aligned} \quad (2.13)$$

The constant  $\frac{1}{4}$  is best possible.

**Proof.** As in [1], we have

$$|\langle x, y \rangle - \langle x, e \rangle \langle e, y \rangle|^2 \leq \left[ \|x\|^2 - |\langle x, e \rangle|^2 \right] \left[ \|y\|^2 - |\langle y, e \rangle|^2 \right], \quad (2.14)$$

$$\begin{aligned} & \|x\|^2 - |\langle x, e \rangle|^2 \\ & = \operatorname{Re} \left[ (\Phi - \langle x, e \rangle) \left( \overline{\langle x, e \rangle} - \overline{\varphi} \right) \right] - \operatorname{Re} \langle \Phi e - x, x - \varphi e \rangle \end{aligned} \quad (2.15)$$



and

$$\begin{aligned} \|y\|^2 - |\langle y, e \rangle|^2 \\ = \operatorname{Re} \left[ (\Gamma - \langle y, e \rangle) \left( \overline{\langle y, e \rangle} - \bar{\gamma} \right) \right] - \operatorname{Re} \langle \Gamma e - x, x - \gamma e \rangle. \end{aligned} \quad (2.16)$$

Using the elementary inequality

$$4 \operatorname{Re} (a\bar{b}) \leq |a + b|^2; \quad a, b \in \mathbb{K} \quad (\mathbb{K} = \mathbb{R}, \mathbb{C}),$$

we may state that

$$\operatorname{Re} \left[ (\Phi - \langle x, e \rangle) \left( \overline{\langle x, e \rangle} - \bar{\varphi} \right) \right] \leq \frac{1}{4} |\Phi - \varphi|^2 \quad (2.17)$$

and

$$\operatorname{Re} \left[ (\Gamma - \langle y, e \rangle) \left( \overline{\langle y, e \rangle} - \bar{\gamma} \right) \right] \leq \frac{1}{4} |\Gamma - \gamma|^2. \quad (2.18)$$

Consequently, by (2.14) – (2.18) we may state that

$$\begin{aligned} |\langle x, y \rangle - \langle x, e \rangle \langle e, y \rangle|^2 \\ \leq \left[ \frac{1}{4} |\Phi - \varphi|^2 - \left( [\operatorname{Re} \langle \Phi e - x, x - \varphi e \rangle]^{\frac{1}{2}} \right)^2 \right] \\ \times \left[ \frac{1}{4} |\Gamma - \gamma|^2 - \left( [\operatorname{Re} \langle \Gamma e - y, y - \gamma e \rangle]^{\frac{1}{2}} \right)^2 \right]. \end{aligned} \quad (2.19)$$

Finally, using the elementary inequality for positive real numbers

$$(m^2 - n^2) (p^2 - q^2) \leq (mp - nq)^2,$$

we have

$$\begin{aligned} & \left[ \frac{1}{4} |\Phi - \varphi|^2 - \left( [\operatorname{Re} \langle \Phi e - x, x - \varphi e \rangle]^{\frac{1}{2}} \right)^2 \right] \\ & \times \left[ \frac{1}{4} |\Gamma - \gamma|^2 - \left( [\operatorname{Re} \langle \Gamma e - y, y - \gamma e \rangle]^{\frac{1}{2}} \right)^2 \right] \\ & \leq \left( \frac{1}{4} |\Phi - \varphi| \cdot |\Gamma - \gamma| - [\operatorname{Re} \langle \Phi e - x, x - \varphi e \rangle]^{\frac{1}{2}} [\operatorname{Re} \langle \Gamma e - y, y - \gamma e \rangle]^{\frac{1}{2}} \right)^2, \end{aligned}$$

giving the desired inequality (2.13). ■

## 2.2.4 Some Companion Inequalities

The following companion of the Grüss inequality in inner product spaces holds [3].

**Theorem 71** Let  $(H, \langle \cdot, \cdot \rangle)$  be an inner product space over  $\mathbb{K}$  ( $\mathbb{K} = \mathbb{R}, \mathbb{C}$ ) and  $e \in H, \|e\| = 1$ . If  $\gamma, \Gamma \in \mathbb{K}$  and  $x, y \in H$  are such that

$$\operatorname{Re} \left\langle \Gamma e - \frac{x+y}{2}, \frac{x+y}{2} - \gamma e \right\rangle \geq 0 \quad (2.20)$$

or equivalently,

$$\left\| \frac{x+y}{2} - \frac{\gamma + \Gamma}{2} \cdot e \right\| \leq \frac{1}{2} |\Gamma - \gamma|,$$

then we have the inequality

$$\operatorname{Re} [\langle x, y \rangle - \langle x, e \rangle \langle e, y \rangle] \leq \frac{1}{4} |\Gamma - \gamma|^2. \quad (2.21)$$

The constant  $\frac{1}{4}$  is best possible in the sense that it cannot be replaced by a smaller constant.

**Proof.** Start with the obvious inequality

$$\operatorname{Re} \langle z, u \rangle \leq \frac{1}{4} \|z + u\|^2; \quad z, u \in H. \quad (2.22)$$

Since

$$\langle x, y \rangle - \langle x, e \rangle \langle e, y \rangle = \langle x - \langle x, e \rangle e, y - \langle y, e \rangle e \rangle,$$

then using (2.22) we may write

$$\begin{aligned} \operatorname{Re} [\langle x, y \rangle - \langle x, e \rangle \langle e, y \rangle] &= \operatorname{Re} [\langle x - \langle x, e \rangle e, y - \langle y, e \rangle e \rangle] \\ &\leq \frac{1}{4} \|x - \langle x, e \rangle e + y - \langle y, e \rangle e\|^2 \\ &= \left\| \frac{x+y}{2} - \left\langle \frac{x+y}{2}, e \right\rangle \cdot e \right\|^2 \\ &= \left\| \frac{x+y}{2} \right\|^2 - \left| \left\langle \frac{x+y}{2}, e \right\rangle \right|^2. \end{aligned} \quad (2.23)$$

If we apply Grüss' inequality in inner product spaces for, say,  $a = b = \frac{x+y}{2}$ , we get

$$\left\| \frac{x+y}{2} \right\|^2 - \left| \left\langle \frac{x+y}{2}, e \right\rangle \right|^2 \leq \frac{1}{4} |\Gamma - \gamma|^2. \quad (2.24)$$

Making use of (2.23) and (2.24) we deduce (2.21).

The fact that  $\frac{1}{4}$  is the best possible constant in (2.21) follows by the fact that if in (2.20) we choose  $x = y$ , then it becomes  $\operatorname{Re} \langle \Gamma e - x, x - \gamma e \rangle \geq 0$ , implying that  $0 \leq \|x\|^2 - |\langle x, e \rangle|^2 \leq \frac{1}{4} |\Gamma - \gamma|^2$ , for which, by Grüss' inequality in inner product spaces, we know that the constant  $\frac{1}{4}$  is best possible. ■

The following corollary might be of interest if one wanted to evaluate the absolute value of

$$\operatorname{Re} [\langle x, y \rangle - \langle x, e \rangle \langle e, y \rangle].$$

**Corollary 72** Let  $(H, \langle \cdot, \cdot \rangle)$  be an inner product space over  $\mathbb{K}$  ( $\mathbb{K} = \mathbb{R}, \mathbb{C}$ ) and  $e \in H, \|e\| = 1$ . If  $\gamma, \Gamma \in \mathbb{K}$  and  $x, y \in H$  are such that

$$\operatorname{Re} \left\langle \Gamma e - \frac{x \pm y}{2}, \frac{x \pm y}{2} - \gamma e \right\rangle \geq 0,$$

or equivalently,

$$\left\| \frac{x \pm y}{2} - \frac{\gamma + \Gamma}{2} \cdot e \right\| \leq \frac{1}{2} |\Gamma - \gamma|,$$

holds, then we have the inequality

$$|\operatorname{Re} [\langle x, y \rangle - \langle x, e \rangle \langle e, y \rangle]| \leq \frac{1}{4} |\Gamma - \gamma|^2. \quad (2.25)$$

If the inner product space  $H$  is real, then (for  $m, M \in \mathbb{R}, M > m$ )

$$\left\langle Me - \frac{x \pm y}{2}, \frac{x \pm y}{2} - me \right\rangle \geq 0,$$

or equivalently,

$$\left\| \frac{x \pm y}{2} - \frac{m + M}{2} \cdot e \right\| \leq \frac{1}{2} (M - m),$$

implies

$$|\langle x, y \rangle - \langle x, e \rangle \langle e, y \rangle| \leq \frac{1}{4} (M - m)^2. \quad (2.26)$$

In both inequalities (2.25) and (2.26), the constant  $\frac{1}{4}$  is best possible.

**Proof.** We only remark that, if

$$\operatorname{Re} \left\langle \Gamma e - \frac{x - y}{2}, \frac{x - y}{2} - \gamma e \right\rangle \geq 0$$

holds, then by Theorem 71, we get

$$\operatorname{Re} [-\langle x, y \rangle + \langle x, e \rangle \langle e, y \rangle] \leq \frac{1}{4} |\Gamma - \gamma|^2,$$

showing that

$$\operatorname{Re} [\langle x, y \rangle - \langle x, e \rangle \langle e, y \rangle] \geq -\frac{1}{4} |\Gamma - \gamma|^2. \quad (2.27)$$

Making use of (2.21) and (2.27) we deduce the desired result (2.25). ■

Finally, we may state and prove the following dual result as well [3].

**Proposition 73** Let  $(H, \langle \cdot, \cdot \rangle)$  be an inner product space over  $\mathbb{K}$  ( $\mathbb{K} = \mathbb{R}, \mathbb{C}$ ) and  $e \in H, \|e\| = 1$ . If  $\varphi, \Phi \in \mathbb{K}$  and  $x, y \in H$  are such that

$$\operatorname{Re} \left[ (\Phi - \langle x, e \rangle) (\overline{\langle x, e \rangle} - \overline{\varphi}) \right] \leq 0, \quad (2.28)$$

then we have the inequalities

$$\begin{aligned} \|x - \langle x, e \rangle e\| &\leq [\operatorname{Re} \langle x - \Phi e, x - \varphi e \rangle]^{\frac{1}{2}} \\ &\leq \frac{\sqrt{2}}{2} \left[ \|x - \Phi e\|^2 + \|x - \varphi e\|^2 \right]^{\frac{1}{2}}. \end{aligned} \quad (2.29)$$

**Proof.** We know that the following identity holds true (see (2.15))

$$\begin{aligned} \|x\|^2 - |\langle x, e \rangle|^2 &= \operatorname{Re} \left[ (\Phi - \langle x, e \rangle) \left( \overline{\langle x, e \rangle} - \overline{\varphi} \right) \right] + \operatorname{Re} \langle x - \Phi e, x - \varphi e \rangle. \end{aligned} \quad (2.30)$$

Using the assumption (2.28) and the fact that

$$\|x\|^2 - |\langle x, e \rangle|^2 = \|x - \langle x, e \rangle e\|^2,$$

by (2.30), we deduce the first inequality in (2.29).

The second inequality in (2.29) follows by the fact that for any  $v, w \in H$  one has

$$\operatorname{Re} \langle w, v \rangle \leq \frac{1}{2} (\|w\|^2 + \|v\|^2).$$

The proposition is thus proved. ■

### 2.2.5 Integral Inequalities

The following proposition holds [3].

**Proposition 74** *If  $f, g, h \in L^2(\Omega, \mathbb{K})$  and  $\varphi, \Phi, \gamma, \Gamma \in \mathbb{K}$ , are such that  $\int_{\Omega} |h(s)|^2 d\mu(s) = 1$  and*

$$\begin{aligned} \int_{\Omega} \operatorname{Re} \left[ (\Phi h(s) - f(s)) \left( \overline{f(s)} - \overline{\varphi h(s)} \right) \right] d\mu(s) &\geq 0, \\ \int_{\Omega} \operatorname{Re} \left[ (\Gamma h(s) - g(s)) \left( \overline{g(s)} - \overline{\gamma h(s)} \right) \right] d\mu(s) &\geq 0 \end{aligned} \quad (2.31)$$

*or, equivalently*

$$\begin{aligned} \left( \int_{\Omega} \left| f(s) - \frac{\Phi + \varphi}{2} h(s) \right|^2 d\mu(s) \right)^{\frac{1}{2}} &\leq \frac{1}{2} |\Phi - \varphi|, \\ \left( \int_{\Omega} \left| g(s) - \frac{\Gamma + \gamma}{2} h(s) \right|^2 d\mu(s) \right)^{\frac{1}{2}} &\leq \frac{1}{2} |\Gamma - \gamma|, \end{aligned}$$

*hold, then we have the following refinement of the Grüss integral inequality*

$$\begin{aligned} &\left| \int_{\Omega} f(s) \overline{g(s)} d\mu(s) - \int_{\Omega} f(s) \overline{h(s)} d\mu(s) \int_{\Omega} h(s) \overline{g(s)} d\mu(s) \right| \\ &\leq \frac{1}{4} |\Phi - \varphi| \cdot |\Gamma - \gamma| - \left[ \int_{\Omega} \operatorname{Re} \left[ (\Phi h(s) - f(s)) \left( \overline{f(s)} - \overline{\varphi h(s)} \right) \right] d\mu(s) \right. \\ &\quad \left. \times \int_{\Omega} \operatorname{Re} \left[ (\Gamma h(s) - g(s)) \left( \overline{g(s)} - \overline{\gamma h(s)} \right) \right] d\mu(s) \right]^{\frac{1}{2}}. \end{aligned}$$

*The constant  $\frac{1}{4}$  is best possible.*

The proof follows by Theorem 70 on choosing  $H = L^2(\Omega, \mathbb{K})$  with the inner product

$$\langle f, g \rangle := \int_{\Omega} f(s) \overline{g(s)} d\mu(s).$$

We omit the details.

**Remark 75** *It is obvious that a sufficient condition for (2.31) to hold is*

$$\operatorname{Re} \left[ (\Phi h(s) - f(s)) \left( \overline{f(s)} - \overline{\varphi h(s)} \right) \right] \geq 0,$$

and

$$\operatorname{Re} \left[ (\Gamma h(s) - g(s)) \left( \overline{g(s)} - \overline{\gamma h(s)} \right) \right] \geq 0,$$

for  $\mu$ -a.e.  $s \in \Omega$ , or equivalently,

$$\begin{aligned} \left| f(s) - \frac{\Phi + \varphi}{2} h(s) \right| &\leq \frac{1}{2} |\Phi - \varphi| |h(s)| \quad \text{and} \\ \left| g(s) - \frac{\Gamma + \gamma}{2} h(s) \right| &\leq \frac{1}{2} |\Gamma - \gamma| |h(s)|, \end{aligned}$$

for  $\mu$ -a.e.  $s \in \Omega$ .

The following result may be stated as well [3].

**Corollary 76** *If  $z, Z, t, T \in \mathbb{K}$ ,  $\mu(\Omega) < \infty$  and  $f, g \in L^2(\Omega, \mathbb{K})$  are such that:*

$$\begin{aligned} \operatorname{Re} \left[ (Z - f(s)) \left( \overline{f(s)} - \bar{z} \right) \right] &\geq 0, \\ \operatorname{Re} \left[ (T - g(s)) \left( \overline{g(s)} - \bar{t} \right) \right] &\geq 0, \quad \text{for a.e. } s \in \Omega, \end{aligned}$$

or, equivalently

$$\begin{aligned} \left| f(s) - \frac{z + Z}{2} \right| &\leq \frac{1}{2} |Z - z|, \\ \left| g(s) - \frac{t + T}{2} \right| &\leq \frac{1}{2} |T - t|, \quad \text{for a.e. } s \in \Omega, \end{aligned}$$

then we have the inequality

$$\begin{aligned} &\left| \frac{1}{\mu(\Omega)} \int_{\Omega} f(s) \overline{g(s)} d\mu(s) - \frac{1}{\mu(\Omega)} \int_{\Omega} f(s) d\mu(s) \cdot \frac{1}{\mu(\Omega)} \int_{\Omega} \overline{g(s)} d\mu(s) \right| \\ &\leq \frac{1}{4} |Z - z| |T - t| - \frac{1}{\mu(\Omega)} \left[ \int_{\Omega} \operatorname{Re} \left[ (Z - f(s)) \left( \overline{f(s)} - \bar{z} \right) \right] d\mu(s) \right. \\ &\quad \left. \times \int_{\Omega} \operatorname{Re} \left[ (T - g(s)) \left( \overline{g(s)} - \bar{t} \right) \right] d\mu(s) \right]^{\frac{1}{2}}. \quad (2.32) \end{aligned}$$

Using Theorem 71 we may state the following result as well [3].

**Proposition 77** *If  $f, g, h \in L^2(\Omega, \mathbb{K})$  and  $\gamma, \Gamma \in \mathbb{K}$  are such that  $\int_{\Omega} |h(s)|^2 d\mu(s) = 1$  and*

$$\int_{\Omega} \operatorname{Re} \left\{ \left[ \Gamma h(s) - \frac{f(s) + g(s)}{2} \right] \times \left[ \frac{\overline{f(s)} + \overline{g(s)}}{2} - \bar{\gamma} \bar{h}(s) \right] \right\} d\mu(s) \geq 0 \quad (2.33)$$

or equivalently,

$$\left( \int_{\Omega} \left| \frac{f(s) + g(s)}{2} - \frac{\gamma + \Gamma}{2} h(s) \right|^2 d\mu(s) \right)^{\frac{1}{2}} \leq \frac{1}{2} |\Gamma - \gamma|, \quad (2.34)$$

holds, then we have the inequality

$$\begin{aligned} I &:= \int_{\Omega} \operatorname{Re} [f(s) \overline{g(s)}] d\mu(s) \\ &\quad - \operatorname{Re} \left[ \int_{\Omega} f(s) \overline{h(s)} d\mu(s) \cdot \int_{\Omega} h(s) \overline{g(s)} d\mu(s) \right] \\ &\leq \frac{1}{4} |\Gamma - \gamma|^2. \end{aligned}$$

If (2.33) and (2.34) hold with “ $\pm$ ” instead of “ $+$ ”, then

$$|I| \leq \frac{1}{4} |\Gamma - \gamma|^2.$$

**Remark 78** *It is obvious that a sufficient condition for (2.33) to hold is*

$$\operatorname{Re} \left\{ \left[ \Gamma h(s) - \frac{f(s) + g(s)}{2} \right] \cdot \left[ \frac{\overline{f(s)} + \overline{g(s)}}{2} - \bar{\gamma} \bar{h}(s) \right] \right\} \geq 0,$$

for a.e.  $s \in \Omega$ , or equivalently,

$$\left| \frac{f(s) + g(s)}{2} - \frac{\gamma + \Gamma}{2} h(s) \right| \leq \frac{1}{2} |\Gamma - \gamma| |h(s)|, \quad \text{for a.e. } s \in \Omega.$$

Finally, the following corollary holds.

**Corollary 79** *If  $Z, z \in \mathbb{K}$ ,  $\mu(\Omega) < \infty$  and  $f, g \in L^2(\Omega, \mathbb{K})$  are such that*

$$\operatorname{Re} \left[ \left( Z - \frac{f(s) + g(s)}{2} \right) \left( \frac{\overline{f(s)} + \overline{g(s)}}{2} - \bar{z} \right) \right] \geq 0, \quad \text{for a.e. } s \in \Omega \quad (2.35)$$

or, equivalently

$$\left| \frac{f(s) + g(s)}{2} - \frac{z + Z}{2} \right| \leq \frac{1}{2} |Z - z|, \quad \text{for a.e. } s \in \Omega, \quad (2.36)$$

then we have the inequality

$$\begin{aligned} J &:= \frac{1}{\mu(\Omega)} \int_{\Omega} \operatorname{Re} \left[ f(s) \overline{g(s)} \right] d\mu(s) \\ &\quad - \operatorname{Re} \left[ \frac{1}{\mu(\Omega)} \int_{\Omega} f(s) d\mu(s) \cdot \frac{1}{\mu(\Omega)} \int_{\Omega} \overline{g(s)} d\mu(s) \right] \\ &\leq \frac{1}{4} |Z - z|^2. \end{aligned}$$

If (2.35) and (2.36) hold with “ $\pm$ ” instead of “ $+$ ”, then

$$|J| \leq \frac{1}{4} |Z - z|^2.$$

**Remark 80** It is obvious that if one chooses the discrete measure above, then all the inequalities in this section may be written for sequences of real or complex numbers. We omit the details.

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## 2.3 Companions of Grüss' Inequality

### 2.3.1 A General Result

The following Grüss type inequality in inner product spaces holds [3].

**Theorem 81** *Let  $x, y, e \in H$  with  $\|e\| = 1$ , and the scalars  $a, A, b, B \in \mathbb{K}$  ( $\mathbb{K} = \mathbb{C}, \mathbb{R}$ ) such that  $\operatorname{Re}(\bar{a}A) > 0$  and  $\operatorname{Re}(\bar{b}B) > 0$ . If*

$$\operatorname{Re} \langle Ae - x, x - ae \rangle \geq 0 \quad \text{and} \quad \operatorname{Re} \langle Be - y, y - be \rangle \geq 0$$

*or equivalently,*

$$\left\| x - \frac{a+A}{2}e \right\| \leq \frac{1}{2}|A-a| \quad \text{and} \quad \left\| y - \frac{b+B}{2}e \right\| \leq \frac{1}{2}|B-b|,$$

*holds, then we have the inequality*

$$|\langle x, y \rangle - \langle x, e \rangle \langle e, y \rangle| \leq \frac{1}{4} M(a, A) M(b, B) |\langle x, e \rangle \langle e, y \rangle|, \quad (2.37)$$

*where  $M(\cdot, \cdot)$  is defined by*

$$M(a, A) := \left[ \frac{(|A| - |a|)^2 + 4[|A\bar{a}| - \operatorname{Re}(A\bar{a})]}{\operatorname{Re}(\bar{a}A)} \right]^{\frac{1}{2}}.$$

*The constant  $\frac{1}{4}$  is best possible in the sense that it cannot be replaced by a smaller constant.*

**Proof.** Start with the inequality

$$|\langle x, y \rangle - \langle x, e \rangle \langle e, y \rangle|^2 \leq \left( \|x\|^2 - |\langle x, e \rangle|^2 \right) \left( \|y\|^2 - |\langle y, e \rangle|^2 \right). \quad (2.38)$$

Now, assume that  $u, v \in H$ , and  $c, C \in \mathbb{K}$  with  $\operatorname{Re}(\bar{c}C) > 0$  and  $\operatorname{Re} \langle Cv - u, u - cv \rangle \geq 0$ . This last inequality is equivalent to

$$\|u\|^2 + \operatorname{Re}(\bar{c}C) \|v\|^2 \leq \operatorname{Re} \left[ C \overline{\langle u, v \rangle} + \bar{c} \langle u, v \rangle \right].$$

Dividing this inequality by  $[\operatorname{Re}(C\bar{c})]^{\frac{1}{2}} > 0$ , we deduce

$$\frac{1}{[\operatorname{Re}(\bar{c}C)]^{\frac{1}{2}}} \|u\|^2 + [\operatorname{Re}(\bar{c}C)]^{\frac{1}{2}} \|v\|^2 \leq \frac{\operatorname{Re} \left[ C \overline{\langle u, v \rangle} + \bar{c} \langle u, v \rangle \right]}{[\operatorname{Re}(\bar{c}C)]^{\frac{1}{2}}}. \quad (2.39)$$

On the other hand, by the elementary inequality

$$\alpha p^2 + \frac{1}{\alpha} q^2 \geq 2pq, \quad \alpha > 0, \quad p, q \geq 0,$$

we deduce

$$2 \|u\| \|v\| \leq \frac{1}{[\operatorname{Re}(\bar{c}C)]^{\frac{1}{2}}} \|u\|^2 + [\operatorname{Re}(\bar{c}C)]^{\frac{1}{2}} \|v\|^2. \quad (2.40)$$

Making use of (2.39) and (2.40) and the fact that for any  $z \in \mathbb{C}$ ,  $\operatorname{Re}(z) \leq |z|$ , we get

$$\|u\| \|v\| \leq \frac{\operatorname{Re}[C\overline{\langle u, v \rangle} + \bar{c}\langle u, v \rangle]}{2[\operatorname{Re}(\bar{c}C)]^{\frac{1}{2}}} \leq \frac{|c| + |C|}{2[\operatorname{Re}(\bar{c}C)]^{\frac{1}{2}}} |\langle u, v \rangle|.$$

Consequently

$$\begin{aligned} \|u\|^2 \|v\|^2 - |\langle u, v \rangle|^2 &\leq \left[ \frac{(|c| + |C|)^2}{4[\operatorname{Re}(\bar{c}C)]} - 1 \right] |\langle u, v \rangle|^2 \\ &= \frac{1}{4} \frac{(|c| - |C|)^2 + 4[|\bar{c}C| - \operatorname{Re}(\bar{c}C)]}{\operatorname{Re}(\bar{c}C)} |\langle u, v \rangle|^2 \\ &= \frac{1}{4} M^2(c, C) |\langle u, v \rangle|^2. \end{aligned} \quad (2.41)$$

Now, if we write (2.41) for the choices  $u = x$ ,  $v = e$  and  $u = y$ ,  $v = e$  respectively and use (2.38), we deduce the desired result (2.37). The sharpness of the constant will be proved in the case where  $H$  is a real inner product space. ■

The following corollary which provides a simpler Grüss type inequality for real constants (and in particular, for real inner product spaces) holds [3].

**Corollary 82** *With the assumptions of Theorem 81 and if  $a, b, A, B \in \mathbb{R}$  are such that  $A > a > 0$ ,  $B > b > 0$  and*

$$\left\| x - \frac{a+A}{2} e \right\| \leq \frac{1}{2} (A-a) \quad \text{and} \quad \left\| y - \frac{b+B}{2} e \right\| \leq \frac{1}{2} (B-b), \quad (2.42)$$

*then we have the inequality*

$$|\langle x, y \rangle - \langle x, e \rangle \langle e, y \rangle| \leq \frac{1}{4} \cdot \frac{(A-a)(B-b)}{\sqrt{abAB}} |\langle x, e \rangle \langle e, y \rangle|. \quad (2.43)$$

*The constant  $\frac{1}{4}$  is best possible.*

**Proof.** To prove the sharpness of the constant  $\frac{1}{4}$  assume that the inequality (2.43) holds in real inner product spaces with  $x = y$  and for a constant  $k > 0$ , i.e.,

$$\|x\|^2 - |\langle x, e \rangle|^2 \leq k \cdot \frac{(A-a)^2}{aA} |\langle x, e \rangle|^2 \quad (A > a > 0), \quad (2.44)$$

provided that  $\left\| x - \frac{a+A}{2} e \right\| \leq \frac{1}{2} (A-a)$ , or equivalently,  $\langle Ae - x, x - ae \rangle \geq 0$ .

We choose  $H = \mathbb{R}^2$ ,  $x = (x_1, x_2) \in \mathbb{R}^2$ ,  $e = \left( \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right)$ . Then we have

$$\begin{aligned} \|x\|^2 - |\langle x, e \rangle|^2 &= x_1^2 + x_2^2 - \frac{(x_1 + x_2)^2}{2} = \frac{(x_1 - x_2)^2}{2}, \\ |\langle x, e \rangle|^2 &= \frac{(x_1 + x_2)^2}{2}, \end{aligned}$$

and by (2.44) we get

$$\frac{(x_1 - x_2)^2}{2} \leq k \cdot \frac{(A - a)^2}{aA} \cdot \frac{(x_1 + x_2)^2}{2}. \quad (2.45)$$

Now, if we let  $x_1 = \frac{a}{\sqrt{2}}$ ,  $x_2 = \frac{A}{\sqrt{2}}$  ( $A > a > 0$ ), then obviously

$$\langle Ae - x, x - ae \rangle = \sum_{i=1}^2 \left( \frac{A}{\sqrt{2}} - x_i \right) \left( x_i - \frac{a}{\sqrt{2}} \right) = 0,$$

which shows that the condition (2.42) is fulfilled, and by (2.45) we get

$$\frac{(A - a)^2}{4} \leq k \cdot \frac{(A - a)^2}{aA} \cdot \frac{(a + A)^2}{4}$$

for any  $A > a > 0$ . This implies

$$(A + a)^2 k \geq aA \quad (2.46)$$

for any  $A > a > 0$ .

Finally, let  $a = 1 - q$ ,  $A = 1 + q$ ,  $q \in (0, 1)$ . Then from (2.46) we get  $4k \geq 1 - q^2$  for any  $q \in (0, 1)$  which produces  $k \geq \frac{1}{4}$ . ■

**Remark 83** If  $\langle x, e \rangle, \langle y, e \rangle$  are assumed not to be zero, then the inequality (2.37) is equivalent to

$$\left| \frac{\langle x, y \rangle}{\langle x, e \rangle \langle e, y \rangle} - 1 \right| \leq \frac{1}{4} M(a, A) M(b, B),$$

while the inequality (2.43) is equivalent to

$$\left| \frac{\langle x, y \rangle}{\langle x, e \rangle \langle e, y \rangle} - 1 \right| \leq \frac{1}{4} \cdot \frac{(A - a)(B - b)}{\sqrt{abAB}}.$$

The constant  $\frac{1}{4}$  is best possible in both inequalities.

### 2.3.2 Some Related Results

The following result holds [3].

**Theorem 84** Let  $(H; \langle \cdot, \cdot \rangle)$  be an inner product space over  $\mathbb{K}$  ( $\mathbb{K} = \mathbb{C}, \mathbb{R}$ ). If  $\gamma, \Gamma \in \mathbb{K}$ ,  $e, x, y \in H$  with  $\|e\| = 1$  and  $\lambda \in (0, 1)$  are such that

$$\operatorname{Re} \langle \Gamma e - (\lambda x + (1 - \lambda)y), (\lambda x + (1 - \lambda)y) - \gamma e \rangle \geq 0, \quad (2.47)$$

or equivalently,

$$\left\| \lambda x + (1 - \lambda)y - \frac{\gamma + \Gamma}{2} e \right\| \leq \frac{1}{2} |\Gamma - \gamma|,$$

then we have the inequality

$$\operatorname{Re} [\langle x, y \rangle - \langle x, e \rangle \langle e, y \rangle] \leq \frac{1}{16} \cdot \frac{1}{\lambda(1-\lambda)} |\Gamma - \gamma|^2. \quad (2.48)$$

The constant  $\frac{1}{16}$  is the best possible constant in (2.48) in the sense that it cannot be replaced by a smaller one.

**Proof.** We know that for any  $z, u \in H$  one has

$$\operatorname{Re} \langle z, u \rangle \leq \frac{1}{4} \|z + u\|^2.$$

Then for any  $a, b \in H$  and  $\lambda \in (0, 1)$  one has

$$\operatorname{Re} \langle a, b \rangle \leq \frac{1}{4\lambda(1-\lambda)} \|\lambda a + (1-\lambda)b\|^2. \quad (2.49)$$

Since

$$\langle x, y \rangle - \langle x, e \rangle \langle e, y \rangle = \langle x - \langle x, e \rangle e, y - \langle y, e \rangle e \rangle \quad (\text{as } \|e\| = 1),$$

using (2.49), we have

$$\begin{aligned} & \operatorname{Re} [\langle x, y \rangle - \langle x, e \rangle \langle e, y \rangle] \\ &= \operatorname{Re} [\langle x - \langle x, e \rangle e, y - \langle y, e \rangle e \rangle] \\ &\leq \frac{1}{4\lambda(1-\lambda)} \|\lambda(x - \langle x, e \rangle e) + (1-\lambda)(y - \langle y, e \rangle e)\|^2 \\ &= \frac{1}{4\lambda(1-\lambda)} \|\lambda x + (1-\lambda)y - \langle \lambda x + (1-\lambda)y, e \rangle e\|^2. \end{aligned} \quad (2.50)$$

Since, for  $m, e \in H$  with  $\|e\| = 1$ , one has the equality

$$\|m - \langle m, e \rangle e\|^2 = \|m\|^2 - |\langle m, e \rangle|^2,$$

then by (2.50) we deduce the inequality

$$\begin{aligned} & \operatorname{Re} [\langle x, y \rangle - \langle x, e \rangle \langle e, y \rangle] \\ &\leq \frac{1}{4\lambda(1-\lambda)} \left[ \|\lambda x + (1-\lambda)y\|^2 - |\langle \lambda x + (1-\lambda)y, e \rangle|^2 \right]. \end{aligned} \quad (2.51)$$

Now, if we apply Grüss' inequality

$$0 \leq \|a\|^2 - |\langle a, e \rangle|^2 \leq \frac{1}{4} |\Gamma - \gamma|^2,$$

provided

$$\operatorname{Re} \langle \Gamma e - a, a - \gamma e \rangle \geq 0,$$

for  $a = \lambda x + (1 - \lambda) y$ , we have

$$\|\lambda x + (1 - \lambda) y\|^2 - |\langle \lambda x + (1 - \lambda) y, e \rangle|^2 \leq \frac{1}{4} |\Gamma - \gamma|^2. \quad (2.52)$$

Utilising (2.51) and (2.52) we deduce the desired inequality (2.48).

To prove the sharpness of the constant  $\frac{1}{16}$ , assume that (2.48) holds with a constant  $C > 0$ , provided that (2.47) is valid, i.e.,

$$\operatorname{Re} [\langle x, y \rangle - \langle x, e \rangle \langle e, y \rangle] \leq C \cdot \frac{1}{\lambda(1 - \lambda)} |\Gamma - \gamma|^2. \quad (2.53)$$

If in (2.53) we choose  $x = y$ , given that (2.47) holds with  $x = y$  and  $\lambda \in (0, 1)$ , then

$$\|x\|^2 - |\langle x, e \rangle|^2 \leq C \cdot \frac{1}{\lambda(1 - \lambda)} |\Gamma - \gamma|^2, \quad (2.54)$$

provided

$$\operatorname{Re} \langle \Gamma e - x, x - \gamma e \rangle \geq 0.$$

Since we know, in Grüss' inequality, that the constant  $\frac{1}{4}$  is best possible, then by (2.54), one has

$$\frac{1}{4} \leq \frac{C}{\lambda(1 - \lambda)} \quad \text{for } \lambda \in (0, 1),$$

giving, for  $\lambda = \frac{1}{2}$ ,  $C \geq \frac{1}{16}$ .

The theorem is completely proved. ■

The following corollary is a natural consequence of the above result [3].

**Corollary 85** *Assume that  $\gamma, \Gamma, e, x, y$  and  $\lambda$  are as in Theorem 84. If*

$$\operatorname{Re} \langle \Gamma e - (\lambda x \pm (1 - \lambda) y), (\lambda x \pm (1 - \lambda) y) - \gamma e \rangle \geq 0,$$

*or equivalently,*

$$\left\| \lambda x \pm (1 - \lambda) y - \frac{\gamma + \Gamma}{2} e \right\| \leq \frac{1}{2} |\Gamma - \gamma|,$$

*then we have the inequality*

$$|\operatorname{Re} [\langle x, y \rangle - \langle x, e \rangle \langle e, y \rangle]| \leq \frac{1}{16} \cdot \frac{1}{\lambda(1 - \lambda)} |\Gamma - \gamma|^2. \quad (2.55)$$

*The constant  $\frac{1}{16}$  is best possible in (2.55).*

**Proof.** Using Theorem 84 for  $(-y)$  instead of  $y$ , we have that

$$\operatorname{Re} \langle \Gamma e - (\lambda x - (1 - \lambda) y), (\lambda x - (1 - \lambda) y) - \gamma e \rangle \geq 0,$$

which implies that

$$\operatorname{Re} [-\langle x, y \rangle + \langle x, e \rangle \langle e, y \rangle] \leq \frac{1}{16} \cdot \frac{1}{\lambda(1 - \lambda)} |\Gamma - \gamma|^2$$

giving

$$\operatorname{Re} [\langle x, y \rangle - \langle x, e \rangle \langle e, y \rangle] \geq -\frac{1}{16} \cdot \frac{1}{\lambda(1-\lambda)} |\Gamma - \gamma|^2. \quad (2.56)$$

Consequently, by (2.48) and (2.56) we deduce the desired inequality (2.55). ■

**Remark 86** If  $M, m \in \mathbb{R}$  with  $M > m$  and, for  $\lambda \in (0, 1)$ ,

$$\left\| \lambda x + (1-\lambda)y - \frac{M+m}{2}e \right\| \leq \frac{1}{2}(M-m), \quad (2.57)$$

then

$$\langle x, y \rangle - \langle x, e \rangle \langle e, y \rangle \leq \frac{1}{16} \cdot \frac{1}{\lambda(1-\lambda)} (M-m)^2.$$

If (2.57) holds with “ $\pm$ ” instead of “ $+$ ”, then

$$|\langle x, y \rangle - \langle x, e \rangle \langle e, y \rangle| \leq \frac{1}{16} \cdot \frac{1}{\lambda(1-\lambda)} (M-m)^2.$$

**Remark 87** If  $\lambda = \frac{1}{2}$  in (2.47), then we obtain the result from [2], i.e.,

$$\operatorname{Re} \left\langle \Gamma e - \frac{x+y}{2}, \frac{x+y}{2} - \gamma e \right\rangle \geq 0,$$

or equivalently,

$$\left\| \frac{x+y}{2} - \frac{\gamma+\Gamma}{2}e \right\| \leq \frac{1}{2} |\Gamma - \gamma|,$$

implies

$$\operatorname{Re} [\langle x, y \rangle - \langle x, e \rangle \langle e, y \rangle] \leq \frac{1}{4} |\Gamma - \gamma|^2. \quad (2.58)$$

The constant  $\frac{1}{4}$  is best possible in (2.58).

For  $\lambda = \frac{1}{2}$ , Corollary 85 and Remark 86 will produce the corresponding results obtained in [2]. We omit the details.

### 2.3.3 Integral Inequalities

The following proposition holds [3].

**Proposition 88** If  $f, g, h \in L^2(\Omega, \mathbb{K})$  and  $\varphi, \Phi, \gamma, \Gamma \in \mathbb{K}$ , are such that  $\operatorname{Re}(\Phi \overline{\varphi}) > 0$ ,  $\operatorname{Re}(\Gamma \overline{\gamma}) > 0$ ,  $\int_{\Omega} |h(s)|^2 d\mu(s) = 1$  and

$$\begin{aligned} \int_{\Omega} \operatorname{Re} \left[ (\Phi h(s) - f(s)) \left( \overline{f(s)} - \overline{\varphi h(s)} \right) \right] d\mu(s) &\geq 0, \\ \int_{\Omega} \operatorname{Re} \left[ (\Gamma h(s) - g(s)) \left( \overline{g(s)} - \overline{\gamma h(s)} \right) \right] d\mu(s) &\geq 0, \end{aligned} \quad (2.59)$$

or equivalently,

$$\begin{aligned} \left( \int_{\Omega} \left| f(s) - \frac{\Phi + \varphi}{2} h(s) \right|^2 d\mu(s) \right)^{\frac{1}{2}} &\leq \frac{1}{2} |\Phi - \varphi|, \\ \left( \int_{\Omega} \left| g(s) - \frac{\Gamma + \gamma}{2} h(s) \right|^2 d\mu(s) \right)^{\frac{1}{2}} &\leq \frac{1}{2} |\Gamma - \gamma|, \end{aligned}$$

then we have the following Grüss type integral inequality

$$\begin{aligned} &\left| \int_{\Omega} f(s) \overline{g(s)} d\mu(s) - \int_{\Omega} f(s) \overline{h(s)} d\mu(s) \int_{\Omega} h(s) \overline{g(s)} d\mu(s) \right| \\ &\leq \frac{1}{4} M(\varphi, \Phi) M(\gamma, \Gamma) \left| \int_{\Omega} f(s) \overline{h(s)} d\mu(s) \int_{\Omega} h(s) \overline{g(s)} d\mu(s) \right|, \quad (2.60) \end{aligned}$$

where

$$M(\varphi, \Phi) := \left[ \frac{(|\Phi| - |\varphi|)^2 + 4[|\Phi\overline{\varphi}| - \operatorname{Re}(\Phi\overline{\varphi})]}{\operatorname{Re}(\Phi\overline{\varphi})} \right]^{\frac{1}{2}}.$$

The constant  $\frac{1}{4}$  is best possible.

The proof follows by Theorem 84 on choosing  $H = L^2(\Omega, \mathbb{K})$  with the inner product

$$\langle f, g \rangle := \int_{\Omega} f(s) \overline{g(s)} d\mu(s).$$

We omit the details.

**Remark 89** It is obvious that a sufficient condition for (2.59) to hold is

$$\operatorname{Re} \left[ (\Phi h(s) - f(s)) (\overline{f(s)} - \overline{\varphi h(s)}) \right] \geq 0,$$

and

$$\operatorname{Re} \left[ (\Gamma h(s) - g(s)) (\overline{g(s)} - \overline{\gamma h(s)}) \right] \geq 0,$$

for  $\mu$ -a.e.  $s \in \Omega$ , or equivalently,

$$\left| f(s) - \frac{\Phi + \varphi}{2} h(s) \right| \leq \frac{1}{2} |\Phi - \varphi| |h(s)|$$

and

$$\left| g(s) - \frac{\Gamma + \gamma}{2} h(s) \right| \leq \frac{1}{2} |\Gamma - \gamma| |h(s)|,$$

for  $\mu$ -a.e.  $s \in \Omega$ .

The following result may be stated as well.

**Corollary 90** *If  $z, Z, t, T \in \mathbb{K}$ ,  $\mu(\Omega) < \infty$  and  $f, g \in L^2(\Omega, \mathbb{K})$  are such that:*

$$\begin{aligned} \operatorname{Re} \left[ (Z - f(s)) (\overline{f(s)} - \bar{z}) \right] &\geq 0, \\ \operatorname{Re} \left[ (T - g(s)) (\overline{g(s)} - \bar{t}) \right] &\geq 0, \quad \text{for a.e. } s \in \Omega \end{aligned}$$

*or equivalently,*

$$\begin{aligned} \left| f(s) - \frac{z+Z}{2} \right| &\leq \frac{1}{2} |Z - z|, \\ \left| g(s) - \frac{t+T}{2} \right| &\leq \frac{1}{2} |T - t|, \quad \text{for a.e. } s \in \Omega; \end{aligned}$$

*then we have the inequality*

$$\begin{aligned} &\left| \frac{1}{\mu(\Omega)} \int_{\Omega} f(s) \overline{g(s)} d\mu(s) - \frac{1}{\mu(\Omega)} \int_{\Omega} f(s) d\mu(s) \cdot \frac{1}{\mu(\Omega)} \int_{\Omega} \overline{g(s)} d\mu(s) \right| \\ &\leq \frac{1}{4} M(z, Z) M(t, T) \left| \frac{1}{\mu(\Omega)} \int_{\Omega} f(s) d\mu(s) \cdot \frac{1}{\mu(\Omega)} \int_{\Omega} \overline{g(s)} d\mu(s) \right|. \quad (2.61) \end{aligned}$$

**Remark 91** *The case of real functions incorporates the following interesting inequality*

$$\left| \frac{\mu(\Omega) \int_{\Omega} f(s) g(s) d\mu(s)}{\int_{\Omega} f(s) d\mu(s) \int_{\Omega} g(s) d\mu(s)} - 1 \right| \leq \frac{1}{4} \cdot \frac{(Z - z)(T - t)}{\sqrt{ztZT}},$$

*provided  $\mu(\Omega) < \infty$ ,*

$$z \leq f(s) \leq Z, \quad t \leq g(s) \leq T$$

*for  $\mu$ -a.e.  $s \in \Omega$ , where  $z, t, Z, T$  are real numbers and the integrals at the denominator are not zero. Here the constant  $\frac{1}{4}$  is best possible in the sense mentioned above.*

Using Theorem 84 we may state the following result as well [3].

**Proposition 92** *If  $f, g, h \in L^2(\Omega, \mathbb{K})$  and  $\gamma, \Gamma \in \mathbb{K}$  are such that  $\int_{\Omega} |h(s)|^2 d\mu(s) = 1$  and*

$$\begin{aligned} &\int_{\Omega} \{ \operatorname{Re} [\Gamma h(s) - (\lambda f(s) + (1 - \lambda) g(s))] \\ &\quad \times [\overline{\lambda f(s) + (1 - \lambda) g(s)} - \bar{\gamma} \bar{h}(s)] \} d\mu(s) \geq 0, \quad (2.62) \end{aligned}$$

*or equivalently,*

$$\left( \int_{\Omega} \left| \lambda f(s) + (1 - \lambda) g(s) - \frac{\gamma + \Gamma}{2} h(s) \right|^2 d\mu(s) \right)^{\frac{1}{2}} \leq \frac{1}{2} |\Gamma - \gamma|, \quad (2.63)$$



then we have the inequality

$$\begin{aligned} I &:= \int_{\Omega} \operatorname{Re} \left[ f(s) \overline{g(s)} \right] d\mu(s) \\ &\quad - \operatorname{Re} \left[ \int_{\Omega} f(s) \overline{h(s)} d\mu(s) \cdot \int_{\Omega} h(s) \overline{g(s)} d\mu(s) \right] \\ &\leq \frac{1}{16} \cdot \frac{1}{\lambda(1-\lambda)} |\Gamma - \gamma|^2. \end{aligned}$$

The constant  $\frac{1}{16}$  is best possible.

If (2.62) and (2.63) hold with “ $\pm$ ” instead of “ $+$ ” (see Corollary 85), then

$$|I| \leq \frac{1}{16} \cdot \frac{1}{\lambda(1-\lambda)} |\Gamma - \gamma|^2.$$

**Remark 93** It is obvious that a sufficient condition for (2.62) to hold is

$$\operatorname{Re} \left\{ [\Gamma h(s) - (\lambda f(s) + (1-\lambda)g(s))] \cdot [\lambda \overline{f(s)} + (1-\lambda)\overline{g(s)} - \bar{\gamma} \bar{h}(s)] \right\} \geq 0$$

for a.e.  $s \in \Omega$ , or equivalently,

$$\left| \lambda f(s) + (1-\lambda)g(s) - \frac{\gamma + \Gamma}{2} h(s) \right| \leq \frac{1}{2} |\Gamma - \gamma| |h(s)|,$$

for a.e.  $s \in \Omega$ .

Finally, the following corollary holds.

**Corollary 94** If  $Z, z \in \mathbb{K}$ ,  $\mu(\Omega) < \infty$  and  $f, g \in L^2(\Omega, \mathbb{K})$  are such that

$$\operatorname{Re} \left[ (Z - (\lambda f(s) + (1-\lambda)g(s))) \left( \lambda \overline{f(s)} + (1-\lambda)\overline{g(s)} - \bar{z} \right) \right] \geq 0 \quad (2.64)$$

for a.e.  $s \in \Omega$ , or, equivalently

$$\left| \lambda f(s) + (1-\lambda)g(s) - \frac{z + Z}{2} \right| \leq \frac{1}{2} |Z - z|, \quad (2.65)$$

for a.e.  $s \in \Omega$ , then we have the inequality

$$\begin{aligned} J &:= \frac{1}{\mu(\Omega)} \int_{\Omega} \operatorname{Re} \left[ f(s) \overline{g(s)} \right] d\mu(s) \\ &\quad - \operatorname{Re} \left[ \frac{1}{\mu(\Omega)} \int_{\Omega} f(s) d\mu(s) \cdot \frac{1}{\mu(\Omega)} \int_{\Omega} \overline{g(s)} d\mu(s) \right] \\ &\leq \frac{1}{16} \cdot \frac{1}{\lambda(1-\lambda)} |Z - z|^2. \end{aligned}$$

If (2.64) and (2.65) hold with “ $\pm$ ” instead of “ $+$ ”, then

$$|J| \leq \frac{1}{16} \cdot \frac{1}{\lambda(1-\lambda)} |Z - z|^2.$$

**Remark 95** It is obvious that if one chooses the discrete measure above, then all the inequalities in this section may be written for sequences of real or complex numbers. We omit the details.

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## 2.4 Other Grüss Type Inequalities

### 2.4.1 General Results

We may state the following result [5].

**Theorem 96** *Let  $(H; \langle \cdot, \cdot \rangle)$  be an inner product space over the real or complex number field  $\mathbb{K}$  ( $\mathbb{K} = \mathbb{R}, \mathbb{K} = \mathbb{C}$ ) and  $x, y, e \in H$  with  $\|e\| = 1$ . If  $r_1, r_2 \in (0, 1)$  and*

$$\|x - e\| \leq r_1, \quad \|y - e\| \leq r_2,$$

*then we have the inequality*

$$|\langle x, y \rangle - \langle x, e \rangle \langle e, y \rangle| \leq r_1 r_2 \|x\| \|y\|. \quad (2.66)$$

*The inequality (2.66) is sharp in the sense that the constant  $c = 1$  in front of  $r_1 r_2$  cannot be replaced by a smaller constant.*

**Proof.** Start with the inequality

$$|\langle x, y \rangle - \langle x, e \rangle \langle e, y \rangle|^2 \leq \left( \|x\|^2 - |\langle x, e \rangle|^2 \right) \left( \|y\|^2 - |\langle y, e \rangle|^2 \right). \quad (2.67)$$

Using Theorem 37 for  $a = e$ , we may state that

$$\|x\|^2 - |\langle x, e \rangle|^2 \leq r_1^2 \|x\|^2, \quad \|y\|^2 - |\langle y, e \rangle|^2 \leq r_2^2 \|y\|^2. \quad (2.68)$$

Utilizing (2.67) and (2.68), we deduce

$$|\langle x, y \rangle - \langle x, e \rangle \langle e, y \rangle|^2 \leq r_1^2 r_2^2 \|x\|^2 \|y\|^2,$$

which is clearly equivalent to the desired inequality (2.66).

The sharpness of the constant follows by the fact that for  $x = y$ ,  $r_1 = r_2 = r$ , we get from (2.66) that

$$\|x\|^2 - |\langle x, e \rangle|^2 \leq r^2 \|x\|^2, \quad (2.69)$$

provided  $\|e\| = 1$  and  $\|x - e\| \leq r < 1$ . The inequality (2.69) is sharp, as shown in Theorem 37, and the proof is completed. ■

Another companion of the Grüss inequality may be stated as well [5].

**Theorem 97** *Let  $(H; \langle \cdot, \cdot \rangle)$  be an inner product space over  $\mathbb{K}$  and  $x, y, e \in H$  with  $\|e\| = 1$ . Suppose also that  $a, A, b, B \in \mathbb{K}$  ( $\mathbb{K} = \mathbb{R}, \mathbb{C}$ ) such that  $\operatorname{Re}(A\bar{a})$ ,  $\operatorname{Re}(B\bar{b}) > 0$ . If either*

$$\operatorname{Re} \langle Ae - x, x - ae \rangle \geq 0, \quad \operatorname{Re} \langle Be - y, y - be \rangle \geq 0,$$

*or equivalently,*

$$\left\| x - \frac{a + A}{2} e \right\| \leq \frac{1}{2} |A - a|, \quad \left\| y - \frac{b + B}{2} e \right\| \leq \frac{1}{2} |B - b|,$$

holds, then we have the inequality

$$|\langle x, y \rangle - \langle x, e \rangle \langle e, y \rangle| \leq \frac{1}{4} \cdot \frac{|A - a| |B - b|}{\sqrt{\operatorname{Re}(A\bar{a}) \operatorname{Re}(B\bar{b})}} |\langle x, e \rangle \langle e, y \rangle|. \quad (2.70)$$

The constant  $\frac{1}{4}$  is best possible.

**Proof.** We know, that

$$|\langle x, y \rangle - \langle x, e \rangle \langle e, y \rangle|^2 \leq \left( \|x\|^2 - |\langle x, e \rangle|^2 \right) \left( \|y\|^2 - |\langle y, e \rangle|^2 \right). \quad (2.71)$$

If we use Corollary 40, then we may state that

$$\|x\|^2 - |\langle x, e \rangle|^2 \leq \frac{1}{4} \cdot \frac{|A - a|^2}{\operatorname{Re}(A\bar{a})} |\langle x, e \rangle|^2 \quad (2.72)$$

and

$$\|y\|^2 - |\langle y, e \rangle|^2 \leq \frac{1}{4} \cdot \frac{|B - b|^2}{\operatorname{Re}(B\bar{b})} |\langle y, e \rangle|^2. \quad (2.73)$$

Utilizing (2.71) – (2.73), we deduce

$$|\langle x, y \rangle - \langle x, e \rangle \langle e, y \rangle|^2 \leq \frac{1}{16} \cdot \frac{|A - a|^2 |B - b|^2}{\operatorname{Re}(A\bar{a}) \operatorname{Re}(B\bar{b})} |\langle x, e \rangle \langle e, y \rangle|^2,$$

which is clearly equivalent to the desired inequality (2.70).

The sharpness of the constant follows from Corollary 40, and we omit the details. ■

**Remark 98** With the assumptions of Theorem 97 and if  $\langle x, e \rangle, \langle y, e \rangle \neq 0$  (that is actually the interesting case), then one has the inequality

$$\left| \frac{\langle x, y \rangle}{\langle x, e \rangle \langle e, y \rangle} - 1 \right| \leq \frac{1}{4} \cdot \frac{|A - a| |B - b|}{\sqrt{\operatorname{Re}(A\bar{a}) \operatorname{Re}(B\bar{b})}}.$$

The constant  $\frac{1}{4}$  is best possible.

We may state the following result [6].

**Theorem 99** Let  $(H; \langle \cdot, \cdot \rangle)$  be an inner product space over the real or complex number field  $\mathbb{K}$  and  $x, y, e \in H$  with  $\|e\| = 1$ . If  $r_1, r_2 > 0$  and

$$\|x - e\| \leq r_1, \quad \|y - e\| \leq r_2,$$

then we have the inequalities

$$\begin{aligned} |\langle x, y \rangle - \langle x, e \rangle \langle e, y \rangle| &\leq \frac{1}{2} r_1 r_2 \sqrt{\|x\| + |\langle x, e \rangle|} \cdot \sqrt{\|y\| + |\langle y, e \rangle|} \\ &\leq r_1 r_2 \|x\| \|y\|. \end{aligned} \quad (2.74)$$

The constant  $\frac{1}{2}$  is best possible in the sense that it cannot be replaced by a smaller constant.

**Proof.** Start with the inequality

$$|\langle x, y \rangle - \langle x, e \rangle \langle e, y \rangle|^2 \leq \left( \|x\|^2 - |\langle x, e \rangle|^2 \right) \left( \|y\|^2 - |\langle y, e \rangle|^2 \right). \quad (2.75)$$

Using Theorem 47 for  $a = e$ , we have

$$\begin{aligned} 0 &\leq \|x\|^2 - |\langle x, e \rangle|^2 \\ &= (\|x\| - |\langle x, e \rangle|) (\|x\| + |\langle x, e \rangle|) \\ &\leq \frac{1}{2} r_1^2 (\|x\| + |\langle x, e \rangle|) \leq r_1^2 \|x\|, \end{aligned} \quad (2.76)$$

and, in a similar way

$$\begin{aligned} 0 &\leq \|y\|^2 - |\langle y, e \rangle|^2 \\ &\leq \frac{1}{2} r_2^2 (\|y\| + |\langle y, e \rangle|) \leq r_2^2 \|y\|. \end{aligned} \quad (2.77)$$

Utilising (2.75) – (2.77), we may state that

$$\begin{aligned} |\langle x, y \rangle - \langle x, e \rangle \langle e, y \rangle|^2 &\leq \frac{1}{4} r_1^2 r_2^2 (\|x\| + |\langle x, e \rangle|) (\|y\| + |\langle y, e \rangle|) \\ &\leq r_1^2 r_2^2 \|x\| \|y\|, \end{aligned} \quad (2.78)$$

giving the desired inequality (2.74).

To prove the sharpness of the constant  $\frac{1}{2}$ , let us assume that  $x = y$  in (2.74), to get

$$\|x\|^2 - |\langle x, e \rangle|^2 \leq \frac{1}{2} r_1^2 (\|x\| + |\langle x, e \rangle|), \quad (2.79)$$

provided  $\|x - e\| \leq r_1$ . If  $x \neq 0$ , then dividing (2.79) with  $\|x\| + |\langle x, e \rangle| > 0$  we get

$$\|x\| - |\langle x, e \rangle| \leq \frac{1}{2} r_1^2 \quad (2.80)$$

provided  $\|x - e\| \leq r_1$ ,  $\|e\| = 1$ . However, (2.80) is in fact (1.74) for  $a = e$ , for which we have shown that  $\frac{1}{2}$  is the best possible constant. ■

The following result also holds [6].

**Theorem 100** *With the assumptions of Theorem 99, we have the inequality*

$$|\langle x, y \rangle - \langle x, e \rangle \langle e, y \rangle| \leq r_1 r_2 \sqrt{\frac{1}{4} r_1^2 + |\langle x, e \rangle|} \cdot \sqrt{\frac{1}{4} r_2^2 + |\langle y, e \rangle|}. \quad (2.81)$$

**Proof.** Note that, from Theorem 48, we have

$$\|x\| \|a\| \leq |\langle x, a \rangle| + \frac{1}{2} r^2, \quad (2.82)$$

provided  $\|x - a\| \leq r$ .

Taking the square of (2.82) and re-arranging the terms, we obtain:

$$0 \leq \|x\|^2 \|a\|^2 - |\langle x, a \rangle|^2 \leq r^2 \left( \frac{1}{4} r^2 + |\langle x, a \rangle| \right),$$

provided  $\|x - a\| \leq r$ .

Using the assumption of the theorem, we then have

$$0 \leq \|x\|^2 - |\langle x, e \rangle|^2 \leq r_1^2 \left( \frac{1}{4} r_1^2 + |\langle x, e \rangle| \right), \quad (2.83)$$

and

$$0 \leq \|y\|^2 - |\langle y, e \rangle|^2 \leq r_2^2 \left( \frac{1}{4} r_2^2 + |\langle y, e \rangle| \right). \quad (2.84)$$

Utilising (2.75), (2.83) and (2.84), we deduce the desired inequality (2.81). ■

The following result may be stated as well [6].

**Theorem 101** *Let  $(H; \langle \cdot, \cdot \rangle)$  be an inner product space over  $\mathbb{K}$  and  $x, y, e \in H$  with  $\|e\| = 1$ . Suppose also that  $a, A, b, B \in \mathbb{K}$  ( $\mathbb{K} = \mathbb{C}, \mathbb{R}$ ) such that  $A \neq -a, B \neq -b$ . If either*

$$\operatorname{Re} \langle Ae - x, x - ae \rangle \geq 0, \quad \operatorname{Re} \langle Be - y, y - be \rangle \geq 0,$$

*or equivalently,*

$$\left\| x - \frac{a+A}{2} e \right\| \leq \frac{1}{2} |A-a|, \quad \left\| y - \frac{b+B}{2} e \right\| \leq \frac{1}{2} |B-b|,$$

*holds, then we have the inequality*

$$\begin{aligned} & |\langle x, y \rangle - \langle x, e \rangle \langle e, y \rangle| \\ & \leq \frac{1}{4} \cdot \frac{|A-a| |B-b|}{\sqrt{|A+a|} \sqrt{|B+b|}} \sqrt{\|x\| + |\langle x, e \rangle|} \cdot \sqrt{\|y\| + |\langle y, e \rangle|} \\ & \leq \frac{1}{2} \cdot \frac{|A-a| |B-b|}{\sqrt{|A+a|} \sqrt{|B+b|}} \sqrt{\|x\| \|y\|}. \end{aligned} \quad (2.85)$$

*The constant  $\frac{1}{4}$  is best possible in (2.85).*

**Proof.** From Theorem 48, we may state that

$$\begin{aligned} 0 & \leq \|x\|^2 - |\langle x, e \rangle|^2 \\ & = (\|x\| - |\langle x, e \rangle|) (\|x\| + |\langle x, e \rangle|) \\ & \leq \frac{1}{4} \cdot \frac{|A-a|^2}{|A+a|} (\|x\| + |\langle x, e \rangle|), \end{aligned} \quad (2.86)$$

and

$$0 \leq \|y\|^2 - |\langle y, e \rangle|^2 \leq \frac{1}{4} \cdot \frac{|B-b|^2}{|B+b|} (\|y\| + |\langle y, e \rangle|). \quad (2.87)$$

Making use of (2.75) and (2.86), (2.87), we deduce the first inequality in (2.85).

The best constant follows by the use of Theorem 48, and we omit the details.

■

Finally, we may state the following theorem as well [6].

**Theorem 102** *With the assumptions of Theorem 101, we have the inequality*

$$\begin{aligned} & |\langle x, y \rangle - \langle x, e \rangle \langle e, y \rangle| \\ & \leq \frac{1}{2} \cdot \frac{|A-a||B-b|}{\sqrt{|A+a||B+b|}} \sqrt{\frac{1}{8} \cdot \frac{|A-a|^2}{|A+a|} + |\langle x, e \rangle|} \\ & \quad \times \sqrt{\frac{1}{8} \cdot \frac{|B-b|^2}{|B+b|} + |\langle y, e \rangle|}. \end{aligned} \quad (2.88)$$

**Proof.** Using Theorem 48, we may state that

$$0 \leq \|x\| - |\langle x, e \rangle| \leq \frac{1}{4} \cdot \frac{|A-a|^2}{|A+a|}.$$

This inequality implies that

$$\|x\|^2 \leq |\langle x, e \rangle|^2 + \frac{1}{2} |\langle x, e \rangle| \cdot \frac{|A-a|^2}{|A+a|} + \frac{1}{16} \cdot \frac{|A-a|^4}{|A+a|^2},$$

giving

$$0 \leq \|x\|^2 - |\langle x, e \rangle|^2 \leq \frac{1}{2} \cdot \frac{|A-a|^2}{|A+a|} \left[ |\langle x, e \rangle| + \frac{1}{8} \cdot \frac{|A-a|^2}{|A+a|} \right]. \quad (2.89)$$

Similarly, we have

$$0 \leq \|y\|^2 - |\langle y, e \rangle|^2 \leq \frac{1}{2} \cdot \frac{|B-b|^2}{|B+b|} \left[ |\langle y, e \rangle| + \frac{1}{8} \cdot \frac{|B-b|^2}{|B+b|} \right]. \quad (2.90)$$

By making use of (2.75) and (2.89), (2.90), we deduce the desired inequality (2.88). ■

For some recent results on Grüss type inequalities in inner product spaces, see [1], [3] and [7].

### 2.4.2 Integral Inequalities

The following Grüss type integral inequality for real or complex-valued functions also holds [5].

**Proposition 103** Let  $f, g, h \in L^2_\rho(\Omega, \mathbb{K})$  with  $\int_\Omega \rho(s) |h(s)|^2 d\mu(s) = 1$  and  $a, A, b, B \in \mathbb{K}$  such that  $\operatorname{Re}(A\bar{a}), \operatorname{Re}(B\bar{b}) > 0$  and

$$\begin{aligned} \operatorname{Re} \left[ (Ah(s) - f(s)) \left( \overline{f(s)} - \overline{a\bar{h}(s)} \right) \right] &\geq 0, \\ \operatorname{Re} \left[ (Bh(s) - g(s)) \left( \overline{g(s)} - \overline{b\bar{h}(s)} \right) \right] &\geq 0, \end{aligned}$$

for  $\mu$ -a.e.  $s \in \Omega$ . Then we have the inequalities

$$\begin{aligned} &\left| \int_\Omega \rho(s) f(s) \overline{g(s)} d\mu(s) - \int_\Omega \rho(s) f(s) \overline{h(s)} d\mu(s) \int_\Omega \rho(s) h(s) \overline{g(s)} d\mu(s) \right| \\ &\leq \frac{1}{4} \cdot \frac{|A-a||B-b|}{\sqrt{\operatorname{Re}(A\bar{a}) \operatorname{Re}(B\bar{b})}} \left| \int_\Omega \rho(s) f(s) \overline{h(s)} d\mu(s) \int_\Omega \rho(s) h(s) \overline{g(s)} d\mu(s) \right|. \end{aligned}$$

The constant  $\frac{1}{4}$  is best possible.

The proof follows by Theorem 97.

By making use of Theorem 101, we may also state

**Proposition 104** Let  $f, g, h \in L^2_\rho(\Omega, \mathbb{K})$  be such that  $\int_\Omega \rho(s) |h(s)|^2 d\mu(s) = 1$ . Suppose also that  $a, A, b, B \in \mathbb{K}$  with  $A \neq -a, B \neq -b$  and

$$\begin{aligned} \operatorname{Re} \left[ (Ah(s) - f(s)) \left( \overline{f(s)} - \overline{a\bar{h}(s)} \right) \right] &\geq 0, \\ \operatorname{Re} \left[ (Bh(s) - g(s)) \left( \overline{g(s)} - \overline{b\bar{h}(s)} \right) \right] &\geq 0, \end{aligned}$$

for  $\mu$ -a.e.  $s \in \Omega$ . Then we have the inequality

$$\begin{aligned} &\left| \int_\Omega \rho(s) f(s) \overline{g(s)} d\mu(s) - \int_\Omega \rho(s) f(s) \overline{h(s)} d\mu(s) \int_\Omega \rho(s) h(s) \overline{g(s)} d\mu(s) \right| \\ &\leq \frac{1}{4} \cdot \frac{|A-a||B-b|}{\sqrt{|A+a||B+b|}} \\ &\quad \times \sqrt{\left( \int_\Omega \rho(s) |f(s)|^2 d\mu(s) \right)^{\frac{1}{2}} + \left| \int_\Omega \rho(s) f(s) \overline{h(s)} d\mu(s) \right|} \\ &\quad \times \sqrt{\left( \int_\Omega \rho(s) |g(s)|^2 d\mu(s) \right)^{\frac{1}{2}} + \left| \int_\Omega \rho(s) g(s) \overline{h(s)} d\mu(s) \right|}. \end{aligned}$$

The constant  $\frac{1}{4}$  is best possible.



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## Chapter 3

# Reverses of Bessel's Inequality

### 3.1 Introduction

Let  $(H, \langle \cdot, \cdot \rangle)$  be an inner product space over  $\mathbb{K}$  ( $\mathbb{K} = \mathbb{R}, \mathbb{C}$ ) and  $\{e_i\}_{i \in I}$  a finite or infinite family of *orthonormal vectors* in  $H$ , i.e.,

$$\langle e_i, e_j \rangle = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases} ; \quad i, j \in I.$$

It is well known that, the following inequality due to Bessel, holds

$$\sum_{i \in I} |\langle x, e_i \rangle|^2 \leq \|x\|^2$$

for any  $x \in H$ , where the meaning of the sum is:

$$\sum_{i \in I} |\langle x, e_i \rangle|^2 := \sup_{F \subset I} \left\{ \sum_{i \in F} |\langle x, e_i \rangle|^2, \quad F \text{ is a finite part of } I \right\}.$$

If  $(H, \langle \cdot, \cdot \rangle)$  is an infinite dimensional Hilbert space and  $\{e_i\}_{i \in \mathbb{N}}$  an orthonormal family in  $H$ , then we also have

$$\sum_{i=0}^{\infty} |\langle x, e_i \rangle|^2 \leq \|x\|^2$$

for any  $x \in H$ . Here the meaning of the series is the usual one.

In this chapter we establish reverses of the Bessel inequality and some Grüss type inequalities for orthonormal families, namely, upper bounds for the expressions

$$\|x\|^2 - \sum_{i \in I} |\langle x, e_i \rangle|^2, \quad \|x\| - \left( \sum_{i \in I} |\langle x, e_i \rangle|^2 \right)^{\frac{1}{2}}, \quad x \in X$$

and

$$\left| \langle x, y \rangle - \sum_{i \in I} \langle x, e_i \rangle \langle e_i, y \rangle \right|, \quad x, y \in H,$$

under various assumptions for the vectors  $x, y$  and the orthonormal family  $\{e_i\}_{i \in I}$ .

## 3.2 Reverses of Bessel's Inequality

### 3.2.1 Introduction

In [1], the author has proved the following Grüss type inequality in real or complex inner product spaces.

**Theorem 105** *Let  $(H, \langle \cdot, \cdot \rangle)$  be an inner product space over  $\mathbb{K}$  ( $\mathbb{K} = \mathbb{R}, \mathbb{C}$ ) and  $e \in H$ ,  $\|e\| = 1$ . If  $\phi, \Phi, \gamma, \Gamma$  are real or complex numbers and  $x, y$  are vectors in  $H$  such that the conditions*

$$\operatorname{Re} \langle \Phi e - x, x - \phi e \rangle \geq 0 \quad \text{and} \quad \operatorname{Re} \langle \Gamma e - y, y - \gamma e \rangle \geq 0 \quad (3.1)$$

*hold, then we have the inequality*

$$|\langle x, y \rangle - \langle x, e \rangle \langle e, y \rangle| \leq \frac{1}{4} |\Phi - \phi| |\Gamma - \gamma|. \quad (3.2)$$

*The constant  $\frac{1}{4}$  is best possible in the sense that it cannot be replaced by a smaller constant.*

In [2], the following refinement of (3.2) has been pointed out.

**Theorem 106** *Let  $H$ ,  $\mathbb{K}$  and  $e$  be as in Theorem 105. If  $\phi, \Phi, \gamma, \Gamma, x, y$  satisfy (3.1) or equivalently*

$$\left\| x - \frac{\phi + \Phi}{2} e \right\| \leq \frac{1}{2} |\Phi - \phi|, \quad \left\| y - \frac{\gamma + \Gamma}{2} e \right\| \leq \frac{1}{2} |\Gamma - \gamma|,$$

*then*

$$\begin{aligned} & |\langle x, y \rangle - \langle x, e \rangle \langle e, y \rangle| \\ & \leq \frac{1}{4} |\Phi - \phi| |\Gamma - \gamma| - [\operatorname{Re} \langle \Phi e - x, x - \phi e \rangle]^{\frac{1}{2}} [\operatorname{Re} \langle \Gamma e - y, y - \gamma e \rangle]^{\frac{1}{2}}. \end{aligned} \quad (3.3)$$

In [4], N. Ujević has generalised Theorem 105 for the case of real inner product spaces as follows.

**Theorem 107** *Let  $(H, \langle \cdot, \cdot \rangle)$  be an inner product space over the real number field  $\mathbb{R}$ , and  $\{e_i\}_{i \in \{1, \dots, n\}}$  an orthonormal family in  $H$ . If  $\phi_i, \gamma_i, \Phi_i, \Gamma_i \in \mathbb{R}$ ,  $i \in \{1, \dots, n\}$  satisfy the condition*

$$\left\langle \sum_{i=1}^n \Phi_i e_i - x, x - \sum_{i=1}^n \phi_i e_i \right\rangle \geq 0, \quad \left\langle \sum_{i=1}^n \Gamma_i e_i - y, y - \sum_{i=1}^n \gamma_i e_i \right\rangle \geq 0,$$

then one has the inequality:

$$\left| \langle x, y \rangle - \sum_{i=1}^n \langle x, e_i \rangle \langle e_i, y \rangle \right| \leq \frac{1}{4} \left[ \sum_{i=1}^n (\Phi_i - \phi_i)^2 \cdot \sum_{i=1}^n (\Gamma_i - \gamma_i)^2 \right]^{\frac{1}{2}}. \quad (3.4)$$

The constant  $\frac{1}{4}$  is best possible in the sense that it cannot be replaced by a smaller constant.

We note that the key point in his proof is the following identity:

$$\begin{aligned} \sum_{i=1}^n (\langle x, e_i \rangle - \phi_i) (\Phi_i - \langle x, e_i \rangle) &= \left\langle x - \sum_{i=1}^n \phi_i e_i, \sum_{i=1}^n \Phi_i e_i - x \right\rangle \\ &= \|x\|^2 - \sum_{i=1}^n \langle x, e_i \rangle^2, \end{aligned}$$

holding for  $x \in H$ ,  $\phi_i, \Phi_i \in \mathbb{R}$ ,  $i \in \{1, \dots, n\}$  and  $\{e_i\}_{i \in \{1, \dots, n\}}$  an orthonormal family of vectors in the real inner product space  $H$ .

In this section, by following [3], we point out a reverse of Bessel's inequality in both real and complex inner product spaces. This result will then be employed to provide a refinement of the Grüss type inequality (3.4) for real or complex inner products. Related results as well as integral inequalities for general measure spaces are also given.

### 3.2.2 A General Result

The following lemma holds [3].

**Lemma 108** *Let  $\{e_i\}_{i \in I}$  be a family of orthonormal vectors in  $H$ ,  $F$  a finite part of  $I$  and  $\phi_i, \Phi_i$  ( $i \in F$ ), real or complex numbers. The following statements are equivalent for  $x \in H$ :*

$$(i) \operatorname{Re} \langle \sum_{i \in F} \Phi_i e_i - x, x - \sum_{i \in F} \phi_i e_i \rangle \geq 0,$$

$$(ii) \left\| x - \sum_{i \in F} \frac{\phi_i + \Phi_i}{2} e_i \right\| \leq \frac{1}{2} \left( \sum_{i \in F} |\Phi_i - \phi_i|^2 \right)^{\frac{1}{2}}.$$

**Proof.** It is easy to see that for  $y, a, A \in H$ , the following are equivalent (see [2, Lemma 1])

$$(b) \operatorname{Re} \langle A - y, y - a \rangle \geq 0 \text{ and}$$

$$(bb) \left\| y - \frac{a+A}{2} \right\| \leq \frac{1}{2} \|A - a\|.$$

Now, for  $a = \sum_{i \in F} \phi_i e_i$ ,  $A = \sum_{i \in F} \Phi_i e_i$ , we have

$$\begin{aligned} \|A - a\| &= \left\| \sum_{i \in F} (\Phi_i - \phi_i) e_i \right\| = \left( \left\| \sum_{i \in F} (\Phi_i - \phi_i) e_i \right\|^2 \right)^{\frac{1}{2}} \\ &= \left( \sum_{i \in F} |\Phi_i - \phi_i|^2 \|e_i\|^2 \right)^{\frac{1}{2}} = \left( \sum_{i \in F} |\Phi_i - \phi_i|^2 \right)^{\frac{1}{2}}, \end{aligned}$$

giving, for  $y = x$ , the desired equivalence. ■

The following reverse of Bessel's inequality holds [3].

**Theorem 109** *Let  $\{e_i\}_{i \in I}$ ,  $F$ ,  $\phi_i, \Phi_i$ ,  $i \in F$  and  $x \in H$  such that either (i) or (ii) of Lemma 108 holds. Then we have the inequality:*

$$\begin{aligned} 0 &\leq \|x\|^2 - \sum_{i \in F} |\langle x, e_i \rangle|^2 \\ &\leq \frac{1}{4} \sum_{i \in F} |\Phi_i - \phi_i|^2 - \operatorname{Re} \left\langle \sum_{i \in F} \Phi_i e_i - x, x - \sum_{i \in F} \phi_i e_i \right\rangle \\ &\leq \frac{1}{4} \sum_{i \in F} |\Phi_i - \phi_i|^2. \end{aligned} \tag{3.5}$$

The constant  $\frac{1}{4}$  is best in both inequalities.

**Proof.** Define

$$I_1 := \sum_{i \in H} \operatorname{Re} \left[ (\Phi_i - \langle x, e_i \rangle) (\overline{\langle x, e_i \rangle} - \overline{\phi_i}) \right]$$

and

$$I_2 := \operatorname{Re} \left[ \left\langle \sum_{i \in H} \Phi_i e_i - x, x - \sum_{i \in H} \phi_i e_i \right\rangle \right].$$

Observe that

$$I_1 = \sum_{i \in H} \operatorname{Re} \left[ \Phi_i \overline{\langle x, e_i \rangle} \right] + \sum_{i \in H} \operatorname{Re} \left[ \overline{\phi_i} \langle x, e_i \rangle \right] - \sum_{i \in H} \operatorname{Re} \left[ \Phi_i \overline{\phi_i} \right] - \sum_{i \in H} |\langle x, e_i \rangle|^2$$

and

$$\begin{aligned} I_2 &= \operatorname{Re} \left[ \sum_{i \in H} \Phi_i \overline{\langle x, e_i \rangle} + \sum_{i \in H} \overline{\phi_i} \langle x, e_i \rangle - \|x\|^2 - \sum_{i \in H} \sum_{j \in H} \Phi_i \overline{\phi_j} \langle e_i, e_j \rangle \right] \\ &= \sum_{i \in H} \operatorname{Re} \left[ \Phi_i \overline{\langle x, e_i \rangle} \right] + \sum_{i \in H} \operatorname{Re} \left[ \overline{\phi_i} \langle x, e_i \rangle \right] - \|x\|^2 - \sum_{i \in H} \operatorname{Re} \left[ \Phi_i \overline{\phi_i} \right]. \end{aligned}$$

Consequently, subtracting  $I_2$  from  $I_1$ , we deduce the following equality that is interesting in its turn

$$\begin{aligned} \|x\|^2 - \sum_{i \in F} |\langle x, e_i \rangle|^2 &= \sum_{i \in H} \operatorname{Re} \left[ (\Phi_i - \langle x, e_i \rangle) (\overline{\langle x, e_i \rangle} - \overline{\phi_i}) \right] \\ &\quad - \operatorname{Re} \left[ \left\langle \sum_{i \in H} \Phi_i e_i - x, x - \sum_{i \in H} \phi_i e_i \right\rangle \right]. \end{aligned} \quad (3.6)$$

Using the following elementary inequality for complex numbers

$$\operatorname{Re} [a\bar{b}] \leq \frac{1}{4} |a + b|^2, \quad a, b \in \mathbb{K},$$

for the choices  $a = \Phi_i - \langle x, e_i \rangle$ ,  $b = \langle x, e_i \rangle - \phi_i$  ( $i \in F$ ), we deduce

$$\sum_{i \in H} \operatorname{Re} \left[ (\Phi_i - \langle x, e_i \rangle) (\overline{\langle x, e_i \rangle} - \overline{\phi_i}) \right] \leq \frac{1}{4} \sum_{i \in H} |\Phi_i - \phi_i|^2. \quad (3.7)$$

Making use of (3.6), (3.7) and the assumption (i), we deduce (3.5).

The sharpness of the constant  $\frac{1}{4}$  was proved for a single element  $e$ ,  $\|e\| = 1$  in [1], or for the real case in [4].

We can give here a simple proof as follows.

Assume that there is a  $c > 0$  such that

$$\begin{aligned} 0 &\leq \|x\|^2 - \sum_{i \in F} |\langle x, e_i \rangle|^2 \\ &\leq c \sum_{i \in F} |\Phi_i - \phi_i|^2 - \operatorname{Re} \left\langle \sum_{i \in F} \Phi_i e_i - x, x - \sum_{i \in F} \phi_i e_i \right\rangle, \end{aligned} \quad (3.8)$$

provided  $\phi_i$ ,  $\Phi_i$ ,  $x$  and  $F$  satisfy (i) or (ii).

We choose  $F = \{1\}$ ,  $e_1 = e_2 = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) \in \mathbb{R}^2$ ,  $x = (x_1, x_2) \in \mathbb{R}^2$ ,  $\Phi_1 = \Phi = m > 0$ ,  $\phi_1 = \phi = -m$ ,  $H = \mathbb{R}^2$  to get from (3.8) that

$$\begin{aligned} 0 &\leq x_1^2 + x_2^2 - \frac{(x_1 + x_2)^2}{2} \\ &\leq 4cm^2 - \left(\frac{m}{\sqrt{2}} - x_1\right) \left(x_1 + \frac{m}{\sqrt{2}}\right) - \left(\frac{m}{\sqrt{2}} - x_2\right) \left(x_2 + \frac{m}{\sqrt{2}}\right), \end{aligned} \quad (3.9)$$

provided

$$\begin{aligned} 0 &\leq \langle me - x, x + me \rangle \\ &= \left(\frac{m}{\sqrt{2}} - x_1\right) \left(x_1 + \frac{m}{\sqrt{2}}\right) + \left(\frac{m}{\sqrt{2}} - x_2\right) \left(x_2 + \frac{m}{\sqrt{2}}\right). \end{aligned} \quad (3.10)$$

If we choose  $x_1 = \frac{m}{\sqrt{2}}$ ,  $x_2 = -\frac{m}{\sqrt{2}}$ , then (3.10) is fulfilled and by (3.9) we get  $m^2 \leq 4cm^2$ , giving  $c \geq \frac{1}{4}$ . ■

### 3.2.3 A Refinement of the Grüss Inequality for Orthonormal Families

The following result holds [3].

**Theorem 110** *Let  $\{e_i\}_{i \in I}$  be a family of orthonormal vectors in  $H$ ,  $F$  a finite part of  $I$  and  $\phi_i, \Phi_i, \gamma_i, \Gamma_i \in \mathbb{K}$ ,  $i \in F$  and  $x, y \in H$ . If either*

$$\begin{aligned} \operatorname{Re} \left\langle \sum_{i \in F} \Phi_i e_i - x, x - \sum_{i \in F} \phi_i e_i \right\rangle &\geq 0, \\ \operatorname{Re} \left\langle \sum_{i \in F} \Gamma_i e_i - y, y - \sum_{i \in F} \gamma_i e_i \right\rangle &\geq 0, \end{aligned}$$

or equivalently,

$$\begin{aligned} \left\| x - \sum_{i \in F} \frac{\Phi_i + \phi_i}{2} e_i \right\| &\leq \frac{1}{2} \left( \sum_{i \in F} |\Phi_i - \phi_i|^2 \right)^{\frac{1}{2}}, \\ \left\| y - \sum_{i \in F} \frac{\Gamma_i + \gamma_i}{2} e_i \right\| &\leq \frac{1}{2} \left( \sum_{i \in F} |\Gamma_i - \gamma_i|^2 \right)^{\frac{1}{2}}, \end{aligned}$$

hold, then we have the inequalities

$$\begin{aligned} &\left| \langle x, y \rangle - \sum_{i \in F} \langle x, e_i \rangle \langle e_i, y \rangle \right| \tag{3.11} \\ &\leq \frac{1}{4} \left( \sum_{i \in F} |\Phi_i - \phi_i|^2 \right)^{\frac{1}{2}} \cdot \left( \sum_{i \in F} |\Gamma_i - \gamma_i|^2 \right)^{\frac{1}{2}} \\ &\quad - \left[ \operatorname{Re} \left\langle \sum_{i \in F} \Phi_i e_i - x, x - \sum_{i \in F} \phi_i e_i \right\rangle \right]^{\frac{1}{2}} \left[ \operatorname{Re} \left\langle \sum_{i \in F} \Gamma_i e_i - y, y - \sum_{i \in F} \gamma_i e_i \right\rangle \right]^{\frac{1}{2}} \\ &\leq \frac{1}{4} \left( \sum_{i \in F} |\Phi_i - \phi_i|^2 \right)^{\frac{1}{2}} \cdot \left( \sum_{i \in F} |\Gamma_i - \gamma_i|^2 \right)^{\frac{1}{2}}. \end{aligned}$$

The constant  $\frac{1}{4}$  is best possible.

**Proof.** Using Schwartz's inequality in the inner product space  $(H, \langle \cdot, \cdot \rangle)$  one has

$$\begin{aligned} &\left| \left\langle x - \sum_{i \in F} \langle x, e_i \rangle e_i, y - \sum_{i \in F} \langle y, e_i \rangle e_i \right\rangle \right|^2 \\ &\leq \left\| x - \sum_{i \in F} \langle x, e_i \rangle e_i \right\|^2 \left\| y - \sum_{i \in F} \langle y, e_i \rangle e_i \right\|^2 \tag{3.12} \end{aligned}$$

and since a simple calculation shows that

$$\left\langle x - \sum_{i \in F} \langle x, e_i \rangle e_i, y - \sum_{i \in F} \langle y, e_i \rangle e_i \right\rangle = \langle x, y \rangle - \sum_{i \in F} \langle x, e_i \rangle \langle e_i, y \rangle$$

and

$$\left\| x - \sum_{i \in F} \langle x, e_i \rangle e_i \right\|^2 = \|x\|^2 - \sum_{i \in F} |\langle x, e_i \rangle|^2$$

for any  $x, y \in H$ , then by (3.12) and by the reverse of Bessel's inequality in Theorem 109, we have

$$\begin{aligned} & \left| \langle x, y \rangle - \sum_{i \in F} \langle x, e_i \rangle \langle e_i, y \rangle \right|^2 \tag{3.13} \\ & \leq \left( \|x\|^2 - \sum_{i \in F} |\langle x, e_i \rangle|^2 \right) \left( \|y\|^2 - \sum_{i \in F} |\langle y, e_i \rangle|^2 \right) \\ & \leq \left[ \frac{1}{4} \sum_{i \in F} |\Phi_i - \phi_i|^2 - \operatorname{Re} \left\langle \sum_{i \in F} \Phi_i e_i - x, x - \sum_{i \in F} \phi_i e_i \right\rangle \right] \\ & \quad \times \left[ \frac{1}{4} \sum_{i \in F} |\Gamma_i - \gamma_i|^2 - \operatorname{Re} \left\langle \sum_{i \in F} \Gamma_i e_i - y, y - \sum_{i \in F} \gamma_i e_i \right\rangle \right] \\ & \leq \left[ \frac{1}{4} \left( \sum_{i \in F} |\Phi_i - \phi_i|^2 \right)^{\frac{1}{2}} \cdot \left( \sum_{i \in F} |\Gamma_i - \gamma_i|^2 \right)^{\frac{1}{2}} \right. \\ & \quad \left. - \left[ \operatorname{Re} \left\langle \sum_{i \in F} \Phi_i e_i - x, x - \sum_{i \in F} \phi_i e_i \right\rangle \right]^{\frac{1}{2}} \left[ \operatorname{Re} \left\langle \sum_{i \in F} \Gamma_i e_i - y, y - \sum_{i \in F} \gamma_i e_i \right\rangle \right]^{\frac{1}{2}} \right] \end{aligned}$$

where, for the last inequality, we have made use of the inequality

$$(m^2 - n^2)(p^2 - q^2) \leq (mp - nq)^2,$$

holding for any  $m, n, p, q > 0$ .

Taking the square root in (3.13) and observing that the quantity in the last square bracket is nonnegative (see for example (3.5)), we deduce the desired result (3.11).

The best constant has been proved in [1] for one element and we omit the details. ■

### 3.2.4 Some Companion Inequalities

The following companion of the Grüss inequality also holds [3].



**Theorem 111** Let  $\{e_i\}_{i \in I}$  be a family of orthornormal vectors in  $H$ ,  $F$  a finite part of  $I$  and  $\phi_i, \Phi_i \in \mathbb{K}$ ,  $i \in F$  and  $x, y \in H$  such that

$$\operatorname{Re} \left\langle \sum_{i \in F} \Phi_i e_i - \frac{x+y}{2}, \frac{x+y}{2} - \sum_{i \in F} \phi_i e_i \right\rangle \geq 0 \quad (3.14)$$

or equivalently,

$$\left\| \frac{x+y}{2} - \sum_{i \in F} \frac{\Phi_i + \phi_i}{2} \cdot e_i \right\| \leq \frac{1}{2} \left( \sum_{i \in F} |\Phi_i - \phi_i|^2 \right)^{\frac{1}{2}},$$

holds, then we have the inequality

$$\operatorname{Re} \left[ \langle x, y \rangle - \sum_{i \in F} \langle x, e_i \rangle \langle e_i, y \rangle \right] \leq \frac{1}{4} \sum_{i \in F} |\Phi_i - \phi_i|^2. \quad (3.15)$$

The constant  $\frac{1}{4}$  is best possible.

**Proof.** Start with the well known inequality

$$\operatorname{Re} \langle z, u \rangle \leq \frac{1}{4} \|z + u\|^2, \quad z, u \in H. \quad (3.16)$$

Since

$$\langle x, y \rangle - \sum_{i \in F} \langle x, e_i \rangle \langle e_i, y \rangle = \left\langle x - \sum_{i \in F} \langle x, e_i \rangle e_i, y - \sum_{i \in F} \langle y, e_i \rangle e_i \right\rangle,$$

for any  $x, y \in H$ , then, by (3.16), we get

$$\begin{aligned} & \operatorname{Re} \left[ \langle x, y \rangle - \sum_{i \in F} \langle x, e_i \rangle \langle e_i, y \rangle \right] \\ &= \operatorname{Re} \left[ \left\langle x - \sum_{i \in F} \langle x, e_i \rangle e_i, y - \sum_{i \in F} \langle y, e_i \rangle e_i \right\rangle \right] \\ &\leq \frac{1}{4} \left\| x - \sum_{i \in F} \langle x, e_i \rangle e_i + y - \sum_{i \in F} \langle y, e_i \rangle e_i \right\|^2 \\ &= \left\| \frac{x+y}{2} - \sum_{i \in F} \left\langle \frac{x+y}{2}, e_i \right\rangle e_i \right\|^2 \\ &= \left\| \frac{x+y}{2} \right\|^2 - \sum_{i \in F} \left| \left\langle \frac{x+y}{2}, e_i \right\rangle \right|^2. \end{aligned} \quad (3.17)$$

If we apply the reverse of Bessel's inequality in Theorem 109 for  $\frac{x+y}{2}$ , we may state that

$$\left\| \frac{x+y}{2} \right\|^2 - \sum_{i \in F} \left| \left\langle \frac{x+y}{2}, e_i \right\rangle \right|^2 \leq \frac{1}{4} \left( \sum_{i \in F} |\Phi_i - \phi_i|^2 \right)^{\frac{1}{2}}. \quad (3.18)$$

Now, by making use of (3.17) and (3.18), we deduce (3.15).

The fact that  $\frac{1}{4}$  is the best constant in (3.15) follows by the fact that if in (3.14) we choose  $x = y$ , then it becomes (i) of Lemma 108, implying (3.5), for which, we have shown that  $\frac{1}{4}$  was the best constant. ■

The following corollary may be of interest if we wish to evaluate the absolute value of

$$\operatorname{Re} \left[ \langle x, y \rangle - \sum_{i \in F} \langle x, e_i \rangle \langle e_i, y \rangle \right].$$

**Corollary 112** *With the assumptions of Theorem 111 and if*

$$\operatorname{Re} \left\langle \sum_{i \in F} \Phi_i e_i - \frac{x \pm y}{2}, \frac{x \pm y}{2} - \sum_{i \in F} \phi_i e_i \right\rangle \geq 0$$

*or equivalently,*

$$\left\| \frac{x \pm y}{2} - \sum_{i \in F} \frac{\Phi_i + \phi_i}{2} \cdot e_i \right\| \leq \frac{1}{2} \left( \sum_{i \in F} |\Phi_i - \phi_i|^2 \right)^{\frac{1}{2}},$$

*holds, then we have the inequality*

$$\left| \operatorname{Re} \left[ \langle x, y \rangle - \sum_{i \in F} \langle x, e_i \rangle \langle e_i, y \rangle \right] \right| \leq \frac{1}{4} \sum_{i \in F} |\Phi_i - \phi_i|^2. \quad (3.19)$$

**Proof.** We only remark that, if

$$\operatorname{Re} \left\langle \sum_{i \in F} \Phi_i e_i - \frac{x - y}{2}, \frac{x - y}{2} - \sum_{i \in F} \phi_i e_i \right\rangle \geq 0$$

holds, then by Theorem 111 for  $(-y)$  instead of  $y$ , we have

$$\operatorname{Re} \left[ -\langle x, y \rangle + \sum_{i \in F} \langle x, e_i \rangle \langle e_i, y \rangle \right] \leq \frac{1}{4} \sum_{i \in F} |\Phi_i - \phi_i|^2,$$

showing that

$$\operatorname{Re} \left[ \langle x, y \rangle - \sum_{i \in F} \langle x, e_i \rangle \langle e_i, y \rangle \right] \geq -\frac{1}{4} \sum_{i \in F} |\Phi_i - \phi_i|^2. \quad (3.20)$$

Making use of (3.15) and (3.20), we deduce the desired inequality (3.19). ■

**Remark 113** If  $H$  is a real inner product space and  $m_i, M_i \in \mathbb{R}$  with the property that

$$\left\langle \sum_{i \in F} M_i e_i - \frac{x \pm y}{2}, \frac{x \pm y}{2} - \sum_{i \in F} m_i e_i \right\rangle \geq 0$$

or equivalently,

$$\left\| \frac{x \pm y}{2} - \sum_{i \in F} \frac{M_i + m_i}{2} \cdot e_i \right\| \leq \frac{1}{2} \left( \sum_{i \in F} (M_i - m_i)^2 \right)^{\frac{1}{2}},$$

then we have the Grüss type inequality

$$\left| \langle x, y \rangle - \sum_{i \in F} \langle x, e_i \rangle \langle e_i, y \rangle \right| \leq \frac{1}{4} \sum_{i \in F} (M_i - m_i)^2.$$

### 3.2.5 Integral Inequalities

Let  $(\Omega, \Sigma, \mu)$  be a measure space consisting of a set  $\Omega$ ,  $\Sigma$  a  $\sigma$ -algebra of parts and  $\mu$  a countably additive and positive measure on  $\Sigma$  with values in  $\mathbb{R} \cup \{\infty\}$ . Let  $\rho \geq 0$  be a  $\mu$ -measurable function on  $\Omega$ . Denote by  $L_\rho^2(\Omega, \mathbb{K})$  the Hilbert space of all real or complex valued functions defined on  $\Omega$  and  $2 - \rho$ -integrable on  $\Omega$ , i.e.,

$$\int_{\Omega} \rho(s) |f(s)|^2 d\mu(s) < \infty.$$

Consider the family  $\{f_i\}_{i \in I}$  of functions in  $L_\rho^2(\Omega, \mathbb{K})$  with the properties that

$$\int_{\Omega} \rho(s) f_i(s) \overline{f_j(s)} d\mu(s) = \delta_{ij}, \quad i, j \in I,$$

where  $\delta_{ij}$  is 0 if  $i \neq j$  and  $\delta_{ij} = 1$  if  $i = j$ .  $\{f_i\}_{i \in I}$  is an orthonormal family in  $L_\rho^2(\Omega, \mathbb{K})$ .

The following proposition holds [3].

**Proposition 114** Let  $\{f_i\}_{i \in I}$  be an orthonormal family of functions in  $L_\rho^2(\Omega, \mathbb{K})$ ,  $F$  a finite subset of  $I$ ,  $\phi_i, \Phi_i \in \mathbb{K}$  ( $i \in F$ ) and  $f \in L_\rho^2(\Omega, \mathbb{K})$ , such that either

$$\begin{aligned} \int_{\Omega} \rho(s) \operatorname{Re} \left[ \left( \sum_{i \in F} \Phi_i f_i(s) - f(s) \right) \right. \\ \left. \times \left( \overline{f}(s) - \sum_{i \in F} \overline{\phi_i} \overline{f_i(s)} \right) \right] d\mu(s) \geq 0 \end{aligned} \quad (3.21)$$

or equivalently,

$$\int_{\Omega} \rho(s) \left| f(s) - \sum_{i \in F} \frac{\Phi_i + \phi_i}{2} f_i(s) \right|^2 d\mu(s) \leq \frac{1}{4} \sum_{i \in F} |\Phi_i - \phi_i|^2.$$

Then we have the inequality

$$\begin{aligned}
0 &\leq \int_{\Omega} \rho(s) |f(s)|^2 d\mu(s) - \sum_{i \in F} \left| \int_{\Omega} \rho(s) f(s) \overline{f_i}(s) d\mu(s) \right|^2 \\
&\leq \frac{1}{4} \sum_{i \in F} |\Phi_i - \phi_i|^2 \\
&\quad - \int_{\Omega} \rho(s) \operatorname{Re} \left[ \left( \sum_{i \in F} \Phi_i f_i(s) - f(s) \right) \left( \overline{f}(s) - \sum_{i \in F} \overline{\phi_i} \overline{f_i}(s) \right) \right] d\mu(s) \\
&\leq \frac{1}{4} \sum_{i \in F} |\Phi_i - \phi_i|^2.
\end{aligned} \tag{3.22}$$

The constant  $\frac{1}{4}$  is best possible in both inequalities.

The proof follows by Theorem 109 applied for the Hilbert space  $L_{\rho}^2(\Omega, \mathbb{K})$  and the orthonormal family  $\{f_i\}_{i \in I}$ .

The following Grüss type inequality also holds [3].

**Proposition 115** *Let  $\{f_i\}_{i \in I}$  and  $F$  be as in Proposition 114. If  $\phi_i, \Phi_i, \gamma_i, \Gamma_i \in \mathbb{K}$  ( $i \in F$ ) and  $f, g \in L_{\rho}^2(\Omega, \mathbb{K})$  so that either*

$$\begin{aligned}
\int_{\Omega} \rho(s) \operatorname{Re} \left[ \left( \sum_{i \in F} \Phi_i f_i(s) - f(s) \right) \left( \overline{f}(s) - \sum_{i \in F} \overline{\phi_i} \overline{f_i}(s) \right) \right] d\mu(s) &\geq 0, \\
\int_{\Omega} \rho(s) \operatorname{Re} \left[ \left( \sum_{i \in F} \Gamma_i f_i(s) - g(s) \right) \left( \overline{g}(s) - \sum_{i \in F} \overline{\gamma_i} \overline{f_i}(s) \right) \right] d\mu(s) &\geq 0,
\end{aligned}$$

or equivalently,

$$\begin{aligned}
\int_{\Omega} \rho(s) \left| f(s) - \sum_{i \in F} \frac{\Phi_i + \phi_i}{2} f_i(s) \right|^2 d\mu(s) &\leq \frac{1}{4} \sum_{i \in F} |\Phi_i - \phi_i|^2, \\
\int_{\Omega} \rho(s) \left| g(s) - \sum_{i \in F} \frac{\Gamma_i + \gamma_i}{2} f_i(s) \right|^2 d\mu(s) &\leq \frac{1}{4} \sum_{i \in F} |\Gamma_i - \gamma_i|^2,
\end{aligned}$$

hold, then we have the inequalities

$$\begin{aligned}
&\left| \int_{\Omega} \rho(s) f(s) g(s) d\mu(s) \right. \\
&\quad \left. - \sum_{i \in F} \int_{\Omega} \rho(s) f(s) \overline{f_i}(s) d\mu(s) \int_{\Omega} \rho(s) f_i(s) \overline{g(s)} d\mu(s) \right| \tag{3.23}
\end{aligned}$$

$$\begin{aligned}
&\leq \frac{1}{4} \left( \sum_{i \in F} |\Phi_i - \phi_i|^2 \right)^{\frac{1}{2}} \left( \sum_{i \in F} |\Gamma_i - \gamma_i|^2 \right)^{\frac{1}{2}} \\
&\quad - \left[ \int_{\Omega} \rho(s) \operatorname{Re} \left[ \left( \sum_{i \in F} \Phi_i f_i(s) - f(s) \right) \left( \bar{f}(s) - \sum_{i \in F} \phi_i \bar{f}_i(s) \right) \right] d\mu(s) \right]^{\frac{1}{2}} \\
&\quad \times \left[ \int_{\Omega} \rho(s) \operatorname{Re} \left[ \left( \sum_{i \in F} \Gamma_i f_i(s) - g(s) \right) \left( \bar{g}(s) - \sum_{i \in F} \bar{\gamma}_i \bar{f}_i(s) \right) \right] d\mu(s) \right]^{\frac{1}{2}} \\
&\leq \frac{1}{4} \left( \sum_{i \in F} |\Phi_i - \phi_i|^2 \right)^{\frac{1}{2}} \left( \sum_{i \in F} |\Gamma_i - \gamma_i|^2 \right)^{\frac{1}{2}}.
\end{aligned}$$

The constant  $\frac{1}{4}$  is the best possible.

The proof follows by Theorem 110 and we omit the details.

**Remark 116** *Similar results may be stated if we apply the other inequalities obtained above. We omit the details.*

In the case of real spaces, the following corollaries provide much simpler sufficient conditions for the reverse of Bessel's inequality (3.22) or for the Grüss type inequality (3.23) to hold.

**Corollary 117** *Let  $\{f_i\}_{i \in I}$  be an orthonormal family of functions in the real Hilbert space  $L^2_{\rho}(\Omega)$ ,  $F$  a finite part of  $I$ ,  $M_i, m_i \in \mathbb{R}$  ( $i \in F$ ) and  $f \in L^2_{\rho}(\Omega)$  such that*

$$\sum_{i \in F} m_i f_i(s) \leq f(s) \leq \sum_{i \in F} M_i f_i(s) \quad \text{for } \mu - \text{a.e. } s \in \Omega.$$

*Then we have the inequalities*

$$\begin{aligned}
0 &\leq \int_{\Omega} \rho(s) f^2(s) d\mu(s) - \sum_{i \in F} \left[ \int_{\Omega} \rho(s) f(s) f_i(s) d\mu(s) \right]^2 \\
&\leq \frac{1}{4} \sum_{i \in F} (M_i - m_i)^2 \\
&\quad - \int_{\Omega} \rho(s) \left( \sum_{i \in F} M_i f_i(s) - f(s) \right) \left( f(s) - \sum_{i \in F} m_i f_i(s) \right) d\mu(s) \\
&\leq \frac{1}{4} \sum_{i \in F} (M_i - m_i)^2.
\end{aligned}$$

The constant  $\frac{1}{4}$  is best possible.

**Corollary 118** *Let  $\{f_i\}_{i \in I}$  and  $F$  be as in Corollary 117. If  $M_i, m_i, N_i, n_i \in \mathbb{R}$  ( $i \in F$ ) and  $f, g \in L^2_\rho(\Omega)$  are such that*

$$\sum_{i \in F} m_i f_i(s) \leq f(s) \leq \sum_{i \in F} M_i f_i(s)$$

and

$$\sum_{i \in F} n_i f_i(s) \leq g(s) \leq \sum_{i \in F} N_i f_i(s), \quad \text{for } \mu - \text{a.e. } s \in \Omega,$$

then we have the inequalities

$$\begin{aligned} & \left| \int_{\Omega} \rho(s) f(s) g(s) d\mu(s) \right. \\ & \quad \left. - \sum_{i \in F} \int_{\Omega} \rho(s) f(s) f_i(s) d\mu(s) \int_{\Omega} \rho(s) g(s) f_i(s) d\mu(s) \right| \\ & \leq \frac{1}{4} \left( \sum_{i \in F} (M_i - m_i)^2 \right)^{\frac{1}{2}} \left( \sum_{i \in F} (N_i - n_i)^2 \right)^{\frac{1}{2}} \\ & \quad - \left[ \int_{\Omega} \rho(s) \left( \sum_{i \in F} M_i f_i(s) - f(s) \right) \left( f(s) - \sum_{i \in F} m_i f_i(s) \right) d\mu(s) \right]^{\frac{1}{2}} \\ & \quad \times \left[ \int_{\Omega} \rho(s) \left( \sum_{i \in F} N_i f_i(s) - g(s) \right) \left( g(s) - \sum_{i \in F} n_i f_i(s) \right) d\mu(s) \right]^{\frac{1}{2}} \\ & \leq \frac{1}{4} \left( \sum_{i \in F} (M_i - m_i)^2 \right)^{\frac{1}{2}} \left( \sum_{i \in F} (N_i - n_i)^2 \right)^{\frac{1}{2}}. \end{aligned}$$

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### 3.3 Another Reverse for Bessel's Inequality

#### 3.3.1 A General Result

The following lemma holds [4].

**Lemma 119** *Let  $\{e_i\}_{i \in I}$  be a family of orthornormal vectors in  $H$ ,  $F$  a finite part of  $I$ ,  $\lambda_i \in \mathbb{K}$ ,  $i \in F$ ,  $r > 0$  and  $x \in H$ . If*

$$\left\| x - \sum_{i \in F} \lambda_i e_i \right\| \leq r,$$

*then we have the inequality*

$$0 \leq \|x\|^2 - \sum_{i \in F} |\langle x, e_i \rangle|^2 \leq r^2 - \sum_{i \in F} |\lambda_i - \langle x, e_i \rangle|^2. \quad (3.24)$$

**Proof.** Consider

$$\begin{aligned} I_1 &:= \left\| x - \sum_{i \in F} \lambda_i e_i \right\|^2 = \left\langle x - \sum_{i \in F} \lambda_i e_i, x - \sum_{j \in F} \lambda_j e_j \right\rangle \\ &= \|x\|^2 - \sum_{i \in F} \lambda_i \overline{\langle x, e_i \rangle} - \sum_{i \in F} \overline{\lambda_i} \langle x, e_i \rangle + \sum_{i \in F} \sum_{j \in F} \lambda_i \overline{\lambda_j} \langle e_i, e_j \rangle \\ &= \|x\|^2 - \sum_{i \in F} \lambda_i \overline{\langle x, e_i \rangle} - \sum_{i \in F} \overline{\lambda_i} \langle x, e_i \rangle + \sum_{i \in F} |\lambda_i|^2 \end{aligned}$$

and

$$\begin{aligned} I_2 &:= \sum_{i \in F} |\lambda_i - \langle x, e_i \rangle|^2 = \sum_{i \in F} (\lambda_i - \langle x, e_i \rangle) (\overline{\lambda_i} - \overline{\langle x, e_i \rangle}) \\ &= \sum_{i \in F} [|\lambda_i|^2 + |\langle x, e_i \rangle|^2 - \overline{\lambda_i} \langle x, e_i \rangle - \lambda_i \overline{\langle x, e_i \rangle}] \\ &= \sum_{i \in F} |\lambda_i|^2 + \sum_{i \in F} |\langle x, e_i \rangle|^2 - \sum_{i \in F} \overline{\lambda_i} \langle x, e_i \rangle - \sum_{i \in F} \lambda_i \overline{\langle x, e_i \rangle}. \end{aligned}$$

If we subtract  $I_2$  from  $I_1$  we deduce the following identity that is interesting in its own right

$$\left\| x - \sum_{i \in F} \lambda_i e_i \right\|^2 - \sum_{i \in F} |\lambda_i - \langle x, e_i \rangle|^2 = \|x\|^2 - \sum_{i \in F} |\langle x, e_i \rangle|^2,$$

from which we easily deduce (3.24). ■

The following reverse of Bessel's inequality holds [4].

**Theorem 120** *Let  $\{e_i\}_{i \in I}$  be a family of orthornormal vectors in  $H$ ,  $F$  a finite part of  $I$ ,  $\phi_i, \Phi_i$ ,  $i \in I$  real or complex numbers. For  $x \in H$ , if either*



$$(i) \operatorname{Re} \langle \sum_{i \in F} \Phi_i e_i - x, x - \sum_{i \in F} \phi_i e_i \rangle \geq 0;$$

or equivalently,

$$(ii) \left\| x - \sum_{i \in F} \frac{\phi_i + \Phi_i}{2} e_i \right\| \leq \frac{1}{2} \left( \sum_{i \in F} |\Phi_i - \phi_i|^2 \right)^{\frac{1}{2}};$$

holds, then the following reverse of Bessel's inequality

$$\begin{aligned} 0 &\leq \|x\|^2 - \sum_{i \in F} |\langle x, e_i \rangle|^2 \\ &\leq \frac{1}{4} \sum_{i \in F} |\Phi_i - \phi_i|^2 - \sum_{i \in F} \left| \frac{\phi_i + \Phi_i}{2} - \langle x, e_i \rangle \right|^2 \\ &\leq \frac{1}{4} \sum_{i \in F} |\Phi_i - \phi_i|^2, \end{aligned} \tag{3.25}$$

is valid.

The constant  $\frac{1}{4}$  is best possible in both inequalities.

**Proof.** If we apply Lemma 119 for  $\lambda_i = \frac{\phi_i + \Phi_i}{2}$  and

$$r := \frac{1}{2} \left( \sum_{i \in F} |\Phi_i - \phi_i|^2 \right)^{\frac{1}{2}},$$

we deduce the first inequality in (3.25).

Let us prove that  $\frac{1}{4}$  is best possible in the second inequality in (3.25).

Assume that there is a  $c > 0$  such that

$$0 \leq \|x\|^2 - \sum_{i \in F} |\langle x, e_i \rangle|^2 \leq c \sum_{i \in F} |\Phi_i - \phi_i|^2 - \sum_{i \in F} \left| \frac{\phi_i + \Phi_i}{2} - \langle x, e_i \rangle \right|^2, \tag{3.26}$$

provided that  $\phi_i, \Phi_i, x$  and  $F$  satisfy (i) and (ii).

We choose  $F = \{1\}$ ,  $e_1 = e = \left( \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right) \in \mathbb{R}^2$ ,  $x = (x_1, x_2) \in \mathbb{R}^2$ ,  $\Phi_1 = \Phi = m > 0$ ,  $\phi_1 = \phi = -m$ ,  $H = \mathbb{R}^2$  to get from (3.26) that

$$\begin{aligned} 0 &\leq x_1^2 + x_2^2 - \frac{(x_1 + x_2)^2}{2} \\ &\leq 4cm^2 - \frac{(x_1 + x_2)^2}{2}, \end{aligned} \tag{3.27}$$

provided

$$\begin{aligned} 0 &\leq \langle me - x, x + me \rangle \\ &= \left( \frac{m}{\sqrt{2}} - x_1 \right) \left( x_1 + \frac{m}{\sqrt{2}} \right) + \left( \frac{m}{\sqrt{2}} - x_2 \right) \left( x_2 + \frac{m}{\sqrt{2}} \right). \end{aligned} \tag{3.28}$$

From (3.27) we get

$$x_1^2 + x_2^2 \leq 4cm^2 \quad (3.29)$$

provided (3.28) holds.

If we choose  $x_1 = \frac{m}{\sqrt{2}}$ ,  $x_2 = -\frac{m}{\sqrt{2}}$ , then (3.28) is fulfilled and by (3.29) we get  $m^2 \leq 4cm^2$ , giving  $c \geq \frac{1}{4}$ . ■

**Remark 121** If  $F = \{1\}$ ,  $e_1 = 1$ ,  $\|e\| = 1$  and for  $\phi, \Phi \in \mathbb{K}$  and  $x \in H$  one has either

$$\operatorname{Re} \langle \Phi e - x, x - \phi e \rangle \geq 0$$

or equivalently,

$$\left\| x - \frac{\phi + \Phi}{2} e \right\| \leq \frac{1}{2} |\Phi - \phi|,$$

then

$$\begin{aligned} 0 &\leq \|x\|^2 - |\langle x, e \rangle|^2 \\ &\leq \frac{1}{4} |\Phi - \phi|^2 - \left| \frac{\phi + \Phi}{2} - \langle x, e \rangle \right|^2 \leq \frac{1}{4} |\Phi - \phi|^2. \end{aligned}$$

The constant  $\frac{1}{4}$  is best possible in both inequalities.

**Remark 122** It is important to compare the bounds provided by Theorem 109 and Theorem 120.

For this purpose, consider

$$B_1(x, e, \phi, \Phi) := \frac{1}{4} (\Phi - \phi)^2 - \langle \Phi e - x, x - \phi e \rangle$$

and

$$B_2(x, e, \phi, \Phi) := \frac{1}{4} (\Phi - \phi)^2 - \left( \frac{\phi + \Phi}{2} - \langle x, e \rangle \right)^2,$$

where  $H$  is a real inner product,  $e \in H$ ,  $\|e\| = 1$ ,  $x \in H$ ,  $\phi, \Phi \in \mathbb{R}$  with

$$\langle \Phi e - x, x - \phi e \rangle \geq 0,$$

or equivalently,

$$\left\| x - \frac{\phi + \Phi}{2} e \right\| \leq \frac{1}{2} |\Phi - \phi|.$$

If we choose  $\phi = -1$ ,  $\Phi = 1$ , then we have

$$B_1(x, e) = 1 - \langle e - x, x + e \rangle = 1 - (\|e\|^2 - \|x\|^2) = \|x\|^2,$$

$$B_2(x, e) = 1 - \langle x, e \rangle^2,$$

provided  $\|x\| \leq 1$ .

Consider  $H = \mathbb{R}^2$ ,  $\langle \mathbf{x}, \mathbf{y} \rangle = x_1 y_1 + x_2 y_2$ ,  $\mathbf{x} = (x_1, x_2)$ ,  $\mathbf{y} = (y_1, y_2) \in \mathbb{R}^2$  and  $\mathbf{e} = \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$ . Then  $\|\mathbf{e}\| = 1$  and we must compare

$$B_1(\mathbf{x}) = x_1^2 + x_2^2$$

with

$$B_2(\mathbf{x}) = 1 - \frac{(x_1 + x_2)^2}{2},$$

provided  $x_1^2 + x_2^2 \leq 1$ .

If we choose  $\mathbf{x}_0 = (1, 0)$ , then  $\|\mathbf{x}_0\| = 1$  and  $B_1(\mathbf{x}_0) = 1$ ,  $B_2(\mathbf{x}_0) = \frac{1}{2}$  showing that  $B_1 > B_2$ . If we choose  $\mathbf{x}_{00} = \left(-\frac{1}{2}, \frac{1}{2}\right)$ , then  $B_1(\mathbf{x}_{00}) = \frac{1}{2}$ ,  $B_2(\mathbf{x}_{00}) = 1$ , showing that  $B_1 < B_2$ .

We may state the following proposition.

**Proposition 123** Let  $\{e_i\}_{i \in I}$  be a family of orthonormal vectors in  $H$ ,  $F$  a finite part of  $I$ ,  $\phi_i, \Phi_i \in \mathbb{K}$  ( $i \in F$ ). If  $x \in H$  either satisfies (i), or equivalently, (ii) of Theorem 120, then the upper bounds

$$B_1(x, e, \phi, \Phi, F) := \frac{1}{4} \sum_{i \in F} |\Phi_i - \phi_i|^2 - \operatorname{Re} \left\langle \sum_{i \in F} \Phi_i e_i - x, x - \sum_{i \in F} \phi_i e_i \right\rangle,$$

$$B_2(x, e, \phi, \Phi, F) := \frac{1}{4} \sum_{i \in F} |\Phi_i - \phi_i|^2 - \sum_{i \in F} \left| \frac{\phi_i + \Phi_i}{2} - \langle x, e_i \rangle \right|^2,$$

for the Bessel's difference  $B_s(x, \mathbf{e}, F) := \|x\|^2 - \sum_{i \in F} |\langle x, e_i \rangle|^2$ , cannot be compared in general.

### 3.3.2 A Refinement of the Grüss Inequality for Orthonormal Families

The following result holds [4].

**Theorem 124** Let  $\{e_i\}_{i \in I}$  be a family of orthonormal vectors in  $H$ ,  $F$  a finite part of  $I$ ,  $\phi_i, \Phi_i, \gamma_i, \Gamma_i \in \mathbb{K}$ ,  $i \in F$  and  $x, y \in H$ . If either

$$\operatorname{Re} \left\langle \sum_{i \in F} \Phi_i e_i - x, x - \sum_{i \in F} \phi_i e_i \right\rangle \geq 0,$$

$$\operatorname{Re} \left\langle \sum_{i \in F} \Gamma_i e_i - y, y - \sum_{i \in F} \gamma_i e_i \right\rangle \geq 0,$$

or equivalently,

$$\begin{aligned}\left\|x - \sum_{i \in F} \frac{\Phi_i + \phi_i}{2} e_i\right\| &\leq \frac{1}{2} \left( \sum_{i \in F} |\Phi_i - \phi_i|^2 \right)^{\frac{1}{2}}, \\ \left\|y - \sum_{i \in F} \frac{\Gamma_i + \gamma_i}{2} e_i\right\| &\leq \frac{1}{2} \left( \sum_{i \in F} |\Gamma_i - \gamma_i|^2 \right)^{\frac{1}{2}},\end{aligned}$$

hold, then we have the inequalities

$$\begin{aligned}0 &\leq \left| \langle x, y \rangle - \sum_{i \in F} \langle x, e_i \rangle \langle e_i, y \rangle \right| \\ &\leq \frac{1}{4} \left( \sum_{i \in F} |\Phi_i - \phi_i|^2 \right)^{\frac{1}{2}} \cdot \left( \sum_{i \in F} |\Gamma_i - \gamma_i|^2 \right)^{\frac{1}{2}} \\ &\quad - \sum_{i \in F} \left| \frac{\Phi_i + \phi_i}{2} - \langle x, e_i \rangle \right| \left| \frac{\Gamma_i + \gamma_i}{2} - \langle y, e_i \rangle \right| \\ &\leq \frac{1}{4} \left( \sum_{i \in F} |\Phi_i - \phi_i|^2 \right)^{\frac{1}{2}} \cdot \left( \sum_{i \in F} |\Gamma_i - \gamma_i|^2 \right)^{\frac{1}{2}}.\end{aligned}\tag{3.30}$$

The constant  $\frac{1}{4}$  is best possible.

**Proof.** Using Schwartz's inequality in the inner product space  $(H, \langle \cdot, \cdot \rangle)$  one has

$$\begin{aligned}\left| \left\langle x - \sum_{i \in F} \langle x, e_i \rangle e_i, y - \sum_{i \in F} \langle y, e_i \rangle e_i \right\rangle \right|^2 \\ \leq \left\| x - \sum_{i \in F} \langle x, e_i \rangle e_i \right\|^2 \left\| y - \sum_{i \in F} \langle y, e_i \rangle e_i \right\|^2\end{aligned}\tag{3.31}$$

and since a simple calculation shows that

$$\left\langle x - \sum_{i \in F} \langle x, e_i \rangle e_i, y - \sum_{i \in F} \langle y, e_i \rangle e_i \right\rangle = \langle x, y \rangle - \sum_{i \in F} \langle x, e_i \rangle \langle e_i, y \rangle$$

and

$$\left\| x - \sum_{i \in F} \langle x, e_i \rangle e_i \right\|^2 = \|x\|^2 - \sum_{i \in F} |\langle x, e_i \rangle|^2$$

for any  $x, y \in H$ , then by (3.31) and by the reverse of Bessel's inequality in Theorem 120, we have

$$\begin{aligned}
& \left| \langle x, y \rangle - \sum_{i \in F} \langle x, e_i \rangle \langle e_i, y \rangle \right|^2 \\
& \leq \left( \|x\|^2 - \sum_{i \in F} |\langle x, e_i \rangle|^2 \right) \left( \|y\|^2 - \sum_{i \in F} |\langle y, e_i \rangle|^2 \right) \\
& \leq \left[ \frac{1}{4} \sum_{i \in F} |\Phi_i - \phi_i|^2 - \sum_{i \in F} \left| \frac{\Phi_i + \phi_i}{2} - \langle x, e_i \rangle \right|^2 \right] \\
& \quad \times \left[ \frac{1}{4} \sum_{i \in F} |\Gamma_i - \gamma_i|^2 - \sum_{i \in F} \left| \frac{\Gamma_i + \gamma_i}{2} - \langle y, e_i \rangle \right|^2 \right] \\
& := K.
\end{aligned} \tag{3.32}$$

Using Aczél's inequality for real numbers, i.e., we recall that

$$\left( a^2 - \sum_{i \in F} a_i^2 \right) \left( b^2 - \sum_{i \in F} b_i^2 \right) \leq \left( ab - \sum_{i \in F} a_i b_i \right)^2, \tag{3.33}$$

provided that  $a, b, a_i, b_i > 0$ ,  $i \in F$ , we may state that

$$\begin{aligned}
K \leq & \left[ \frac{1}{4} \left( \sum_{i \in F} |\Phi_i - \phi_i|^2 \right)^{\frac{1}{2}} \cdot \left( \sum_{i \in F} |\Gamma_i - \gamma_i|^2 \right)^{\frac{1}{2}} \right. \\
& \left. - \sum_{i \in F} \left| \frac{\Phi_i + \phi_i}{2} - \langle x, e_i \rangle \right| \left| \frac{\Gamma_i + \gamma_i}{2} - \langle y, e_i \rangle \right| \right]^2. \tag{3.34}
\end{aligned}$$

Using (3.32) and (3.34) we conclude that

$$\begin{aligned}
\left| \langle x, y \rangle - \sum_{i \in F} \langle x, e_i \rangle \langle e_i, y \rangle \right|^2 \leq & \left[ \frac{1}{4} \left( \sum_{i \in F} |\Phi_i - \phi_i|^2 \right)^{\frac{1}{2}} \cdot \left( \sum_{i \in F} |\Gamma_i - \gamma_i|^2 \right)^{\frac{1}{2}} \right. \\
& \left. - \sum_{i \in F} \left| \frac{\Phi_i + \phi_i}{2} - \langle x, e_i \rangle \right| \left| \frac{\Gamma_i + \gamma_i}{2} - \langle y, e_i \rangle \right| \right]^2. \tag{3.35}
\end{aligned}$$

Taking the square root in (3.35) and taking into account that the quantity in the last square bracket is nonnegative (this follows by (3.25) and by the Cauchy-Bunyakovsky-Schwarz inequality), we deduce the second inequality in (3.30).

The fact that  $\frac{1}{4}$  is the best possible constant follows by Theorem 120 and we omit the details. ■

The following corollary may be stated [4].

**Corollary 125** Let  $e \in H$ ,  $\|e\| = 1$ ,  $\phi, \Phi, \gamma, \Gamma \in \mathbb{K}$  and  $x, y \in H$  are such that either

$$\operatorname{Re} \langle \Phi e - x, x - \phi e \rangle \geq 0 \quad \text{and} \quad \operatorname{Re} \langle \Gamma e - y, y - \gamma e \rangle \geq 0$$

or equivalently,

$$\left\| x - \frac{\phi + \Phi}{2} e \right\| \leq \frac{1}{2} |\Phi - \phi|, \quad \left\| y - \frac{\gamma + \Gamma}{2} e \right\| \leq \frac{1}{2} |\Gamma - \gamma|,$$

hold. Then we have the following refinement of Grüss' inequality

$$\begin{aligned} 0 &\leq |\langle x, y \rangle - \langle x, e \rangle \langle e, y \rangle| \\ &\leq \frac{1}{4} |\Phi - \phi| |\Gamma - \gamma| - \left| \frac{\phi + \Phi}{2} - \langle x, e \rangle \right| \left| \frac{\gamma + \Gamma}{2} - \langle y, e \rangle \right| \\ &\leq \frac{1}{4} |\Phi - \phi| |\Gamma - \gamma|. \end{aligned}$$

The constant  $\frac{1}{4}$  is best possible in both inequalities.

### 3.3.3 Some Companion Inequalities

The following companion of the Grüss inequality also holds [4].

**Theorem 126** Let  $\{e_i\}_{i \in I}$  be a family of orthonormal vectors in  $H$ ,  $F$  a finite part of  $I$  and  $\phi_i, \Phi_i \in \mathbb{K}$ ,  $i \in F$ ,  $x, y \in H$  and  $\lambda \in (0, 1)$ , such that either

$$\operatorname{Re} \left\langle \sum_{i \in F} \Phi_i e_i - (\lambda x + (1 - \lambda) y), \lambda x + (1 - \lambda) y - \sum_{i \in F} \phi_i e_i \right\rangle \geq 0 \quad (3.36)$$

or equivalently,

$$\left\| \lambda x + (1 - \lambda) y - \sum_{i \in F} \frac{\Phi_i + \phi_i}{2} \cdot e_i \right\| \leq \frac{1}{2} \left( \sum_{i \in F} |\Phi_i - \phi_i|^2 \right)^{\frac{1}{2}},$$

holds. Then we have the inequality

$$\begin{aligned} &\operatorname{Re} \left[ \langle x, y \rangle - \sum_{i \in F} \langle x, e_i \rangle \langle e_i, y \rangle \right] \\ &\leq \frac{1}{16} \cdot \frac{1}{\lambda(1 - \lambda)} \sum_{i \in F} |\Phi_i - \phi_i|^2 \\ &\quad - \frac{1}{4} \frac{1}{\lambda(1 - \lambda)} \sum_{i \in F} \left| \frac{\Phi_i + \phi_i}{2} - \langle \lambda x + (1 - \lambda) y, e_i \rangle \right|^2 \\ &\leq \frac{1}{16} \cdot \frac{1}{\lambda(1 - \lambda)} \sum_{i \in F} |\Phi_i - \phi_i|^2. \end{aligned} \quad (3.37)$$

The constant  $\frac{1}{16}$  is the best possible constant in (3.37) in the sense that it cannot be replaced by a smaller constant.

**Proof.** We know that for any  $z, u \in H$ , one has

$$\operatorname{Re} \langle z, u \rangle \leq \frac{1}{4} \|z + u\|^2.$$

Then for any  $a, b \in H$  and  $\lambda \in (0, 1)$  one has

$$\operatorname{Re} \langle a, b \rangle \leq \frac{1}{4\lambda(1-\lambda)} \|\lambda a + (1-\lambda)b\|^2. \quad (3.38)$$

Since

$$\langle x, y \rangle - \sum_{i \in F} \langle x, e_i \rangle \langle e_i, y \rangle = \left\langle x - \sum_{i \in F} \langle x, e_i \rangle e_i, y - \sum_{i \in F} \langle y, e_i \rangle e_i \right\rangle,$$

for any  $x, y \in H$ , then, by (3.38), we get

$$\begin{aligned} & \operatorname{Re} \left[ \langle x, y \rangle - \sum_{i \in F} \langle x, e_i \rangle \langle e_i, y \rangle \right] \\ &= \operatorname{Re} \left[ \left\langle x - \sum_{i \in F} \langle x, e_i \rangle e_i, y - \sum_{i \in F} \langle y, e_i \rangle e_i \right\rangle \right] \\ &\leq \frac{1}{4\lambda(1-\lambda)} \left\| \lambda \left( x - \sum_{i \in F} \langle x, e_i \rangle e_i \right) + (1-\lambda) \left( y - \sum_{i \in F} \langle y, e_i \rangle e_i \right) \right\|^2 \\ &= \frac{1}{4\lambda(1-\lambda)} \left\| \lambda x + (1-\lambda)y - \sum_{i \in F} \langle \lambda x + (1-\lambda)y, e_i \rangle e_i \right\|^2 \\ &= \frac{1}{4\lambda(1-\lambda)} \left[ \|\lambda x + (1-\lambda)y\|^2 - \sum_{i \in F} |\langle \lambda x + (1-\lambda)y, e_i \rangle|^2 \right]. \end{aligned} \quad (3.39)$$

If we apply the reverse of Bessel's inequality in Theorem 120 for  $\lambda x + (1-\lambda)y$ , we may state that

$$\begin{aligned} & \|\lambda x + (1-\lambda)y\|^2 - \sum_{i \in F} |\langle \lambda x + (1-\lambda)y, e_i \rangle|^2 \\ &\leq \frac{1}{4} \sum_{i \in F} |\Phi_i - \phi_i|^2 - \sum_{i \in F} \left| \frac{\Phi_i + \phi_i}{2} - \langle \lambda x + (1-\lambda)y, e_i \rangle \right|^2 \\ &\leq \frac{1}{4} \sum_{i \in F} |\Phi_i - \phi_i|^2. \end{aligned} \quad (3.40)$$

Now, by making use of (3.39) and (3.40), we deduce (3.37).

The fact that  $\frac{1}{16}$  is the best possible constant in (3.37) follows by the fact that if in (3.36) we choose  $x = y$ , then it becomes (i) of Theorem 120, implying for  $\lambda = \frac{1}{2}$  (3.25), for which, we have shown that  $\frac{1}{4}$  was the best constant. ■

**Remark 127** In practical applications we may use only the inequality between the first and the last terms in (3.37).

**Remark 128** If in Theorem 126, we choose  $\lambda = \frac{1}{2}$ , then we get

$$\begin{aligned} & \operatorname{Re} \left[ \langle x, y \rangle - \sum_{i \in F} \langle x, e_i \rangle \langle e_i, y \rangle \right] \\ & \leq \frac{1}{4} \sum_{i \in F} |\Phi_i - \phi_i|^2 - \sum_{i \in F} \left| \frac{\Phi_i + \phi_i}{2} - \left\langle \frac{x+y}{2}, e_i \right\rangle \right|^2 \\ & \leq \frac{1}{4} \sum_{i \in F} |\Phi_i - \phi_i|^2, \end{aligned}$$

provided

$$\operatorname{Re} \left\langle \sum_{i \in F} \Phi_i e_i - \frac{x+y}{2}, \frac{x+y}{2} - \sum_{i \in F} \phi_i e_i \right\rangle \geq 0$$

or equivalently,

$$\left\| \frac{x+y}{2} - \sum_{i \in F} \frac{\Phi_i + \phi_i}{2} \cdot e_i \right\| \leq \frac{1}{2} \left( \sum_{i \in F} |\Phi_i - \phi_i|^2 \right)^{\frac{1}{2}}.$$

**Corollary 129** With the assumptions of Theorem 126 and if

$$\operatorname{Re} \left\langle \sum_{i \in F} \Phi_i e_i - (\lambda x \pm (1-\lambda)y), \lambda x \pm (1-\lambda)y - \sum_{i \in F} \phi_i e_i \right\rangle \geq 0$$

or, equivalently

$$\left\| \lambda x \pm (1-\lambda)y - \sum_{i \in F} \frac{\Phi_i + \phi_i}{2} \cdot e_i \right\| \leq \frac{1}{2} \left( \sum_{i \in F} |\Phi_i - \phi_i|^2 \right)^{\frac{1}{2}},$$

then we have the inequality

$$\left| \operatorname{Re} \left[ \langle x, y \rangle - \sum_{i \in F} \langle x, e_i \rangle \langle e_i, y \rangle \right] \right| \leq \frac{1}{16} \cdot \frac{1}{\lambda(1-\lambda)} \sum_{i \in F} |\Phi_i - \phi_i|^2. \quad (3.41)$$

The constant  $\frac{1}{16}$  is best possible in (3.41).

**Remark 130** If  $H$  is a real inner product space and  $m_i, M_i \in \mathbb{R}$  with the property

$$\left\langle \sum_{i \in F} M_i e_i - (\lambda x \pm (1-\lambda)y), \lambda x \pm (1-\lambda)y - \sum_{i \in F} m_i e_i \right\rangle \geq 0$$



or equivalently,

$$\left\| \lambda x \pm (1 - \lambda) y - \sum_{i \in F} \frac{M_i + m_i}{2} \cdot e_i \right\| \leq \frac{1}{2} \left[ \sum_{i \in F} (M_i - m_i)^2 \right]^{\frac{1}{2}},$$

then we have the Grüss type inequality

$$\left| \langle x, y \rangle - \sum_{i \in F} \langle x, e_i \rangle \langle e_i, y \rangle \right| \leq \frac{1}{16} \cdot \frac{1}{\lambda(1 - \lambda)} \sum_{i \in F} (M_i - m_i)^2.$$

### 3.3.4 Integral Inequalities

The following proposition holds [4].

**Proposition 131** *Let  $\{f_i\}_{i \in I}$  be an orthonormal family of functions in  $L_\rho^2(\Omega, \mathbb{K})$ ,  $F$  a finite subset of  $I$ ,  $\phi_i, \Phi_i \in \mathbb{K}$  ( $i \in F$ ) and  $f \in L_\rho^2(\Omega, \mathbb{K})$ , so that either*

$$\int_\Omega \rho(s) \operatorname{Re} \left[ \left( \sum_{i \in F} \Phi_i f_i(s) - f(s) \right) \left( \overline{f}(s) - \sum_{i \in F} \overline{\phi_i} \overline{f_i}(s) \right) \right] d\mu(s) \geq 0$$

or equivalently,

$$\int_\Omega \rho(s) \left| f(s) - \sum_{i \in F} \frac{\Phi_i + \phi_i}{2} \cdot f_i(s) \right|^2 d\mu(s) \leq \frac{1}{4} \sum_{i \in F} |\Phi_i - \phi_i|^2.$$

Then we have the inequality

$$\begin{aligned} 0 &\leq \int_\Omega \rho(s) |f(s)|^2 d\mu(s) - \sum_{i \in F} \left| \int_\Omega \rho(s) f(s) \overline{f_i}(s) d\mu(s) \right|^2 \\ &\leq \frac{1}{4} \sum_{i \in F} |\Phi_i - \phi_i|^2 - \sum_{i \in F} \left| \frac{\Phi_i + \phi_i}{2} - \int_\Omega \rho(s) f(s) \overline{f_i}(s) d\mu(s) \right|^2 \\ &\leq \frac{1}{4} \sum_{i \in F} |\Phi_i - \phi_i|^2. \end{aligned} \quad (3.42)$$

The constant  $\frac{1}{4}$  is best possible in both inequalities.

The proof follows by Theorem 120 applied for the Hilbert space  $L_\rho^2(\Omega, \mathbb{K})$  and the orthonormal family  $\{f_i\}_{i \in I}$ .

The following Grüss type inequality also holds [4].

**Proposition 132** *Let  $\{f_i\}_{i \in I}$  and  $F$  be as in Proposition 131. If  $\phi_i, \Phi_i, \gamma_i, \Gamma_i \in \mathbb{K}$  ( $i \in F$ ) and  $f, g \in L_\rho^2(\Omega, \mathbb{K})$  so that either*

$$\begin{aligned} \int_\Omega \rho(s) \operatorname{Re} \left[ \left( \sum_{i \in F} \Phi_i f_i(s) - f(s) \right) \left( \overline{f}(s) - \sum_{i \in F} \overline{\phi_i} \overline{f_i}(s) \right) \right] d\mu(s) &\geq 0, \\ \int_\Omega \rho(s) \operatorname{Re} \left[ \left( \sum_{i \in F} \Gamma_i f_i(s) - g(s) \right) \left( \overline{g}(s) - \sum_{i \in F} \overline{\gamma_i} \overline{f_i}(s) \right) \right] d\mu(s) &\geq 0, \end{aligned}$$

or equivalently,

$$\begin{aligned} \int_{\Omega} \rho(s) \left| f(s) - \sum_{i \in F} \frac{\Phi_i + \phi_i}{2} f_i(s) \right|^2 d\mu(s) &\leq \frac{1}{4} \sum_{i \in F} |\Phi_i - \phi_i|^2, \\ \int_{\Omega} \rho(s) \left| g(s) - \sum_{i \in F} \frac{\Gamma_i + \gamma_i}{2} f_i(s) \right|^2 d\mu(s) &\leq \frac{1}{4} \sum_{i \in F} |\Gamma_i - \gamma_i|^2, \end{aligned}$$

hold, then we have the inequalities

$$\begin{aligned} &\left| \int_{\Omega} \rho(s) f(s) \overline{g(s)} d\mu(s) \right. \\ &\quad \left. - \sum_{i \in F} \int_{\Omega} \rho(s) f(s) \overline{f_i(s)} d\mu(s) \int_{\Omega} \rho(s) f_i(s) \overline{g(s)} d\mu(s) \right| \\ &\leq \frac{1}{4} \left( \sum_{i \in F} |\Phi_i - \phi_i|^2 \right)^{\frac{1}{2}} \left( \sum_{i \in F} |\Gamma_i - \gamma_i|^2 \right)^{\frac{1}{2}} \\ &\quad - \sum_{i \in F} \left| \frac{\Phi_i + \phi_i}{2} - \int_{\Omega} \rho(s) f(s) \overline{f_i(s)} d\mu(s) \right| \\ &\quad \times \left| \frac{\Gamma_i + \gamma_i}{2} - \int_{\Omega} \rho(s) g(s) \overline{f_i(s)} d\mu(s) \right| \\ &\leq \frac{1}{4} \left( \sum_{i \in F} |\Phi_i - \phi_i|^2 \right)^{\frac{1}{2}} \left( \sum_{i \in F} |\Gamma_i - \gamma_i|^2 \right)^{\frac{1}{2}}. \end{aligned} \tag{3.43}$$

The constant  $\frac{1}{4}$  is the best possible.

The proof follows by Theorem 124 and we omit the details.

**Remark 133** Similar results may be stated if one applies the inequalities in the above subsections. We omit the details.

In the case of real spaces, the following corollaries provide much simpler sufficient conditions for the reverse of Bessel's inequality (3.42) or for the Grüss type inequality (3.43) to hold.

**Corollary 134** Let  $\{f_i\}_{i \in I}$  be an orthonormal family of functions in the real Hilbert space  $L^2_{\rho}(\Omega)$ ,  $F$  a finite part of  $I$ ,  $M_i, m_i \in \mathbb{R}$  ( $i \in F$ ) and  $f \in L^2_{\rho}(\Omega)$  so that

$$\sum_{i \in F} m_i f_i(s) \leq f(s) \leq \sum_{i \in F} M_i f_i(s) \quad \text{for } \mu - \text{a.e. } s \in \Omega.$$

Then we have the inequalities

$$\begin{aligned}
0 &\leq \int_{\Omega} \rho(s) f^2(s) d\mu(s) - \sum_{i \in F} \left[ \int_{\Omega} \rho(s) f(s) f_i(s) d\mu(s) \right]^2 \\
&\leq \frac{1}{4} \sum_{i \in F} (M_i - m_i)^2 - \sum_{i \in F} \left[ \frac{M_i + m_i}{2} - \int_{\Omega} \rho(s) f(s) f_i(s) d\mu(s) \right]^2 \\
&\leq \frac{1}{4} \sum_{i \in F} (M_i - m_i)^2.
\end{aligned}$$

The constant  $\frac{1}{4}$  is best possible.

**Corollary 135** Let  $\{f_i\}_{i \in I}$  and  $F$  be as in Corollary 134. If  $M_i, m_i, N_i, n_i \in \mathbb{R}$  ( $i \in F$ ) and  $f, g \in L^2_{\rho}(\Omega)$  are such that

$$\sum_{i \in F} m_i f_i(s) \leq f(s) \leq \sum_{i \in F} M_i f_i(s)$$

and

$$\sum_{i \in F} n_i f_i(s) \leq g(s) \leq \sum_{i \in F} N_i f_i(s) \quad \text{for } \mu - a.e. \ s \in \Omega,$$

hold, then we have the inequalities

$$\begin{aligned}
&\left| \int_{\Omega} \rho(s) f(s) g(s) d\mu(s) \right. \\
&\quad \left. - \sum_{i \in F} \int_{\Omega} \rho(s) f(s) f_i(s) d\mu(s) \int_{\Omega} \rho(s) g(s) f_i(s) d\mu(s) \right| \\
&\leq \frac{1}{4} \left( \sum_{i \in F} (M_i - m_i)^2 \right)^{\frac{1}{2}} \left( \sum_{i \in F} (N_i - n_i)^2 \right)^{\frac{1}{2}} \\
&\quad - \sum_{i \in F} \left| \frac{M_i + m_i}{2} - \int_{\Omega} \rho(s) f(s) f_i(s) d\mu(s) \right| \\
&\quad \times \left| \frac{N_i + n_i}{2} - \int_{\Omega} \rho(s) g(s) f_i(s) d\mu(s) \right| \\
&\leq \frac{1}{4} \left( \sum_{i \in F} (M_i - m_i)^2 \right)^{\frac{1}{2}} \left( \sum_{i \in F} (N_i - n_i)^2 \right)^{\frac{1}{2}}.
\end{aligned}$$

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## 3.4 More Reverses of Bessel's Inequality

### 3.4.1 A General Result

The following reverse of Bessel's inequality holds [3].

**Theorem 136** *Let  $\{e_i\}_{i \in I}$  be a family of orthonormal vectors in  $H$ ,  $F$  a finite part of  $I$ , and  $\phi_i, \Phi_i$  ( $i \in F$ ), real or complex numbers such that  $\sum_{i \in F} \operatorname{Re}(\Phi_i \overline{\phi_i}) > 0$ . If  $x \in H$  is such that either*

$$(i) \operatorname{Re} \left\langle \sum_{i \in F} \Phi_i e_i - x, x - \sum_{i \in F} \phi_i e_i \right\rangle \geq 0;$$

*or equivalently,*

$$(ii) \left\| x - \sum_{i \in F} \frac{\phi_i + \Phi_i}{2} e_i \right\| \leq \frac{1}{2} \left( \sum_{i \in F} |\Phi_i - \phi_i|^2 \right)^{\frac{1}{2}};$$

*holds, then one has the inequality*

$$\|x\|^2 \leq \frac{1}{4} \cdot \frac{\sum_{i \in F} (|\Phi_i| + |\phi_i|)^2}{\sum_{i \in F} \operatorname{Re}(\Phi_i \overline{\phi_i})} \sum_{i \in F} |\langle x, e_i \rangle|^2. \quad (3.44)$$

*The constant  $\frac{1}{4}$  is best possible in the sense that it cannot be replaced by a smaller constant.*

**Proof.** Observe that

$$\begin{aligned} \operatorname{Re} \left\langle \sum_{i \in F} \Phi_i e_i - x, x - \sum_{i \in F} \phi_i e_i \right\rangle \\ = \sum_{i \in F} \operatorname{Re} \left[ \Phi_i \overline{\langle x, e_i \rangle} + \overline{\phi_i} \langle x, e_i \rangle \right] - \|x\|^2 - \sum_{i \in F} \operatorname{Re}(\Phi_i \overline{\phi_i}), \end{aligned}$$

giving, from (i), that

$$\|x\|^2 + \sum_{i \in F} \operatorname{Re}(\Phi_i \overline{\phi_i}) \leq \sum_{i \in F} \operatorname{Re} \left[ \Phi_i \overline{\langle x, e_i \rangle} + \overline{\phi_i} \langle x, e_i \rangle \right]. \quad (3.45)$$

On the other hand, by the elementary inequality

$$\alpha p^2 + \frac{1}{\alpha} q^2 \geq 2pq, \quad \alpha > 0, \quad p, q \geq 0;$$

we deduce

$$2\|x\| \leq \frac{\|x\|^2}{\left[ \sum_{i \in F} \operatorname{Re}(\Phi_i \overline{\phi_i}) \right]^{\frac{1}{2}}} + \left[ \sum_{i \in F} \operatorname{Re}(\Phi_i \overline{\phi_i}) \right]^{\frac{1}{2}}. \quad (3.46)$$

Dividing (3.45) by  $\left[ \sum_{i \in F} \operatorname{Re}(\Phi_i \overline{\phi_i}) \right]^{\frac{1}{2}} > 0$  and using (3.46), we obtain

$$\|x\| \leq \frac{1}{2} \frac{\sum_{i \in F} \operatorname{Re} \left[ \Phi_i \overline{\langle x, e_i \rangle} + \overline{\phi_i} \langle x, e_i \rangle \right]}{\left[ \sum_{i \in F} \operatorname{Re}(\Phi_i \overline{\phi_i}) \right]^{\frac{1}{2}}}, \quad (3.47)$$

which is also an interesting inequality in itself.

Using the Cauchy-Bunyakovsky-Schwarz inequality for real numbers, we get

$$\begin{aligned} \sum_{i \in F} \operatorname{Re} \left[ \Phi_i \overline{\langle x, e_i \rangle} + \overline{\phi_i} \langle x, e_i \rangle \right] &\leq \sum_{i \in F} \left| \Phi_i \overline{\langle x, e_i \rangle} + \overline{\phi_i} \langle x, e_i \rangle \right| \\ &\leq \sum_{i \in F} (|\Phi_i| + |\phi_i|) |\langle x, e_i \rangle| \\ &\leq \left[ \sum_{i \in F} (|\Phi_i| + |\phi_i|)^2 \right]^{\frac{1}{2}} \left[ \sum_{i \in F} |\langle x, e_i \rangle|^2 \right]^{\frac{1}{2}}. \end{aligned} \quad (3.48)$$

Making use of (3.47) and (3.48), we deduce the desired result (3.44).

To prove the sharpness of the constant  $\frac{1}{4}$ , let us assume that (3.44) holds with a constant  $c > 0$ , i.e.,

$$\|x\|^2 \leq c \cdot \frac{\sum_{i \in F} (|\Phi_i| + |\phi_i|)^2}{\sum_{i \in F} \operatorname{Re} (\Phi_i \overline{\phi_i})} \sum_{i \in F} |\langle x, e_i \rangle|^2, \quad (3.49)$$

provided  $x, \phi_i, \Phi_i, i \in F$  satisfies (i).

Choose  $F = \{1\}$ ,  $e_1 = e$ ,  $\|e\| = 1$ ,  $\phi_i = m$ ,  $\Phi_i = M$  with  $m, M > 0$ , then, by (3.49), we get

$$\|x\|^2 \leq c \frac{(M + m)^2}{mM} |\langle x, e \rangle|^2 \quad (3.50)$$

provided

$$\operatorname{Re} \langle Me - x, x - me \rangle \geq 0. \quad (3.51)$$

If  $x = me$ , then obviously (3.51) holds, and by (3.50) we get

$$m^2 \leq c \frac{(M + m)^2}{mM} m^2$$

giving  $mM \leq c(M + m)^2$  for  $m, M > 0$ . Now, if in this inequality we choose  $m = 1 - \varepsilon$ ,  $M = 1 + \varepsilon$  ( $\varepsilon \in (0, 1)$ ), then we get  $1 - \varepsilon^2 \leq 4c$  for  $\varepsilon \in (0, 1)$ , from where we deduce  $c \geq \frac{1}{4}$ . ■

**Remark 137** By the use of (3.47), the second inequality in (3.48) and the Hölder inequality, we may state the following reverses of Bessel's inequality as

well:

$$\|x\|^2 \leq \frac{1}{2} \cdot \frac{1}{[\sum_{i \in F} \operatorname{Re}(\Phi_i \overline{\phi_i})]^{\frac{1}{2}}} \times \begin{cases} \max_{i \in F} \{|\Phi_i| + |\phi_i|\} \sum_{i \in F} |\langle x, e_i \rangle|; \\ \left[ \sum_{i \in F} (|\Phi_i| + |\phi_i|)^p \right]^{\frac{1}{p}} \left( \sum_{i \in F} |\langle x, e_i \rangle|^q \right)^{\frac{1}{q}}, \\ \text{for } p > 1, \frac{1}{p} + \frac{1}{q} = 1; \\ \max_{i \in F} |\langle x, e_i \rangle| \sum_{i \in F} [|\Phi_i| + |\phi_i|]. \end{cases}$$

The following corollary holds [3].

**Corollary 138** *With the assumption of Theorem 136 and if either (i) or (ii) holds, then*

$$0 \leq \|x\|^2 - \sum_{i \in F} |\langle x, e_i \rangle|^2 \leq \frac{1}{4} M^2(\Phi, \phi, F) \sum_{i \in F} |\langle x, e_i \rangle|^2, \quad (3.52)$$

where

$$M(\Phi, \phi, F) := \left[ \frac{\sum_{i \in F} \left\{ (|\Phi_i| - |\phi_i|)^2 + 4 [|\Phi_i \overline{\phi_i}| - \operatorname{Re}(\Phi_i \overline{\phi_i})] \right\}}{\sum_{i \in F} \operatorname{Re}(\Phi_i \overline{\phi_i})} \right]^{\frac{1}{2}}.$$

The constant  $\frac{1}{4}$  is best possible.

**Proof.** The inequality (3.52) follows by (3.44) on subtracting the same quantity  $\sum_{i \in F} |\langle x, e_i \rangle|^2$  from both sides.

To prove the sharpness of the constant  $\frac{1}{4}$ , assume that (3.52) holds with  $c > 0$ , i.e.,

$$0 \leq \|x\|^2 - \sum_{i \in F} |\langle x, e_i \rangle|^2 \leq c M^2(\Phi, \phi, F) \sum_{i \in F} |\langle x, e_i \rangle|^2 \quad (3.53)$$

provided the condition (i) holds.

Choose  $F = \{1\}$ ,  $e_1 = e$ ,  $\|e\| = 1$ ,  $\phi_i = \phi$ ,  $\Phi_i = \Phi$ ,  $\phi, \Phi > 0$  in (3.53) to get

$$0 \leq \|x\|^2 - |\langle x, e \rangle|^2 \leq c \frac{(\Phi - \phi)^2}{\phi \Phi} |\langle x, e \rangle|^2, \quad (3.54)$$

provided

$$\langle \Phi e - x, x - \phi e \rangle \geq 0. \quad (3.55)$$

If  $H = \mathbb{R}^2$ ,  $x = (x_1, x_2) \in \mathbb{R}^2$ ,  $e = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$  then we have

$$\begin{aligned}\|x\|^2 - |\langle x, e \rangle|^2 &= x_1^2 + x_2^2 - \frac{(x_1 + x_2)^2}{2} = \frac{1}{2}(x_1 - x_2)^2, \\ |\langle x, e \rangle|^2 &= \frac{(x_1 + x_2)^2}{2}\end{aligned}$$

and by (3.54) we get

$$\frac{(x_1 - x_2)^2}{2} \leq c \frac{(\Phi - \phi)^2}{\phi\Phi} \cdot \frac{(x_1 + x_2)^2}{2}. \quad (3.56)$$

Now, if we let  $x_1 = \frac{\phi}{\sqrt{2}}$ ,  $x_2 = \frac{\Phi}{\sqrt{2}}$  ( $\phi, \Phi > 0$ ) then obviously

$$\langle \Phi e - x, x - \phi e \rangle = \sum_{i=1}^2 \left( \frac{\Phi}{\sqrt{2}} - x_i \right) \left( x_i - \frac{\phi}{\sqrt{2}} \right) = 0,$$

which shows that (3.55) is fulfilled, and thus by (3.56) we obtain

$$\frac{(\Phi - \phi)^2}{4} \leq c \frac{(\Phi - \phi)^2}{\phi\Phi} \cdot \frac{(\Phi + \phi)^2}{4}$$

for any  $\Phi > \phi > 0$ . This implies

$$c(\Phi + \phi)^2 \geq \phi\Phi \quad (3.57)$$

for any  $\Phi > \phi > 0$ .

Finally, let  $\phi = 1 - \varepsilon$ ,  $\Phi = 1 + \varepsilon$ ,  $\varepsilon \in (0, 1)$ . Then from (3.57) we get  $4c \geq 1 - \varepsilon^2$  for any  $\varepsilon \in (0, 1)$  which produces  $c \geq \frac{1}{4}$ . ■

**Remark 139** If  $\{e_i\}_{i \in I}$  is an orthonormal family in the real inner product space  $(H; \langle \cdot, \cdot \rangle)$  and  $M_i, m_i \in \mathbb{R}$ ,  $i \in F$  ( $F$  is a finite part of  $I$ ) and  $x \in H$  are such that  $M_i, m_i \geq 0$  for  $i \in F$  with  $\sum_{i \in F} M_i m_i \geq 0$  and

$$\left\langle \sum_{i \in F} M_i e_i - x, x - \sum_{i \in F} m_i e_i \right\rangle \geq 0,$$

then we have the inequality

$$0 \leq \|x\|^2 - \sum_{i \in F} [\langle x, e_i \rangle]^2 \leq \frac{1}{4} \cdot \frac{\sum_{i \in F} (M_i - m_i)^2}{\sum_{i \in F} M_i m_i} \cdot \sum_{i \in F} [\langle x, e_i \rangle]^2.$$

The constant  $\frac{1}{4}$  is best possible.

The following reverse of the Schwarz's inequality in inner product spaces holds.



**Corollary 140** Let  $x, y \in H$  and  $\delta, \Delta \in \mathbb{K}$  ( $\mathbb{K} = \mathbb{C}, \mathbb{R}$ ) with the property that  $\operatorname{Re}(\Delta\bar{\delta}) > 0$ . If either

$$\operatorname{Re} \langle \Delta y - x, x - \delta y \rangle \geq 0,$$

or equivalently,

$$\left\| x - \frac{\delta + \Delta}{2} \cdot y \right\| \leq \frac{1}{2} |\Delta - \delta| \|y\|,$$

holds, then we have the inequalities

$$\begin{aligned} \|x\| \|y\| &\leq \frac{1}{2} \cdot \frac{\operatorname{Re} [\Delta \overline{\langle x, y \rangle} + \bar{\delta} \langle x, y \rangle]}{\sqrt{\Delta \bar{\delta}}} \\ &\leq \frac{1}{2} \cdot \frac{|\Delta| + |\delta|}{\sqrt{\Delta \bar{\delta}}} |\langle x, y \rangle|, \end{aligned} \quad (3.58)$$

$$\begin{aligned} 0 &\leq \|x\| \|y\| - |\langle x, y \rangle| \\ &\leq \frac{1}{2} \cdot \frac{\left( \sqrt{|\Delta|} - \sqrt{|\delta|} \right)^2 + 2 \left( \sqrt{\Delta \bar{\delta}} - \sqrt{\operatorname{Re}(\Delta \bar{\delta})} \right)}{\sqrt{\Delta \bar{\delta}}} |\langle x, y \rangle|, \end{aligned} \quad (3.59)$$

$$\|x\|^2 \|y\|^2 \leq \frac{1}{4} \cdot \frac{(|\Delta| + |\delta|)^2}{\operatorname{Re}(\Delta \bar{\delta})} |\langle x, y \rangle|^2, \quad (3.60)$$

and

$$\begin{aligned} 0 &\leq \|x\|^2 \|y\|^2 - |\langle x, y \rangle|^2 \\ &\leq \frac{1}{4} \cdot \frac{(|\Delta| + |\delta|)^2 + 4(|\Delta \bar{\delta}| - \operatorname{Re}(\Delta \bar{\delta}))}{\operatorname{Re}(\Delta \bar{\delta})} |\langle x, y \rangle|^2. \end{aligned} \quad (3.61)$$

The constants  $\frac{1}{2}$  and  $\frac{1}{4}$  are best possible.

**Proof.** The inequality (3.58) follows from (3.47) on choosing  $F = \{1\}$ ,  $e_1 = e = \frac{y}{\|y\|}$ ,  $\Phi_1 = \Phi = \Delta \|y\|$ ,  $\phi_1 = \phi = \delta \|y\|$  ( $y \neq 0$ ). The inequality (3.59) is equivalent with (3.58). The inequality (3.60) follows from (3.44) for  $F = \{1\}$  and the same choices as above. Finally, (3.61) is obviously equivalent with (3.60). ■

### 3.4.2 Some Grüss Type Inequalities for Orthonormal Families

The following result holds [3].

**Theorem 141** Let  $\{e_i\}_{i \in I}$  be a family of orthornormal vectors in  $H$ ,  $F$  a finite part of  $I$ ,  $\phi_i, \Phi_i, \gamma_i, \Gamma_i \in \mathbb{K}$ ,  $i \in F$  and  $x, y \in H$ . If either

$$\begin{aligned} \operatorname{Re} \left\langle \sum_{i \in F} \Phi_i e_i - x, x - \sum_{i \in F} \phi_i e_i \right\rangle &\geq 0, \\ \operatorname{Re} \left\langle \sum_{i \in F} \Gamma_i e_i - y, y - \sum_{i \in F} \gamma_i e_i \right\rangle &\geq 0, \end{aligned}$$

or equivalently,

$$\begin{aligned} \left\| x - \sum_{i \in F} \frac{\Phi_i + \phi_i}{2} e_i \right\| &\leq \frac{1}{2} \left( \sum_{i \in F} |\Phi_i - \phi_i|^2 \right)^{\frac{1}{2}}, \\ \left\| y - \sum_{i \in F} \frac{\Gamma_i + \gamma_i}{2} e_i \right\| &\leq \frac{1}{2} \left( \sum_{i \in F} |\Gamma_i - \gamma_i|^2 \right)^{\frac{1}{2}}, \end{aligned}$$

hold, then we have the inequality

$$\begin{aligned} 0 &\leq \left| \langle x, y \rangle - \sum_{i \in F} \langle x, e_i \rangle \langle e_i, y \rangle \right| \\ &\leq \frac{1}{4} M(\Phi, \phi, F) M(\Gamma, \gamma, F) \left( \sum_{i \in F} |\langle x, e_i \rangle|^2 \right)^{\frac{1}{2}} \left( \sum_{i \in F} |\langle y, e_i \rangle|^2 \right)^{\frac{1}{2}}, \end{aligned} \quad (3.62)$$

where  $M(\Phi, \phi, F)$  is defined in Corollary 138.

The constant  $\frac{1}{4}$  is best possible.

**Proof.** By the reverse of Bessel's inequality in Corollary 138, we have

$$\begin{aligned} &\left| \langle x, y \rangle - \sum_{i \in F} \langle x, e_i \rangle \langle e_i, y \rangle \right|^2 \\ &\leq \left( \|x\|^2 - \sum_{i \in F} |\langle x, e_i \rangle|^2 \right) \left( \|y\|^2 - \sum_{i \in F} |\langle y, e_i \rangle|^2 \right) \\ &\leq \frac{1}{4} M^2(\Phi, \phi, F) \sum_{i \in F} |\langle x, e_i \rangle|^2 \cdot \frac{1}{4} M^2(\Gamma, \gamma, F) \sum_{i \in F} |\langle y, e_i \rangle|^2. \end{aligned} \quad (3.63)$$

Taking the square root in (3.63), we deduce (3.62).

The fact that  $\frac{1}{4}$  is the best possible constant follows by Corollary 138 and we omit the details. ■

The following corollary for real inner product spaces holds [3].

**Corollary 142** Let  $\{e_i\}_{i \in I}$  be a family of orthornormal vectors in  $H$ ,  $F$  a finite part of  $I$ ,  $M_i, m_i, N_i, n_i \geq 0$ ,  $i \in F$  and  $x, y \in H$  such that  $\sum_{i \in F} M_i m_i > 0$ ,  $\sum_{i \in F} N_i n_i > 0$  and

$$\left\langle \sum_{i \in F} M_i e_i - x, x - \sum_{i \in F} m_i e_i \right\rangle \geq 0, \quad \left\langle \sum_{i \in F} N_i e_i - y, y - \sum_{i \in F} n_i e_i \right\rangle \geq 0.$$

Then we have the inequality

$$\begin{aligned} 0 &\leq \left| \langle x, y \rangle - \sum_{i \in F} \langle x, e_i \rangle \langle y, e_i \rangle \right|^2 \\ &\leq \frac{1}{16} \cdot \frac{\sum_{i \in F} (M_i - m_i)^2 \sum_{i \in F} (N_i - n_i)^2 \sum_{i \in F} |\langle x, e_i \rangle|^2 \sum_{i \in F} |\langle y, e_i \rangle|^2}{\sum_{i \in F} M_i m_i \sum_{i \in F} N_i n_i}. \end{aligned}$$

The constant  $\frac{1}{16}$  is best possible.

In the case where the family  $\{e_i\}_{i \in I}$  reduces to a single vector, we may deduce from Theorem 141 the following particular case first obtained in [4].

**Corollary 143** Let  $e \in H$ ,  $\|e\| = 1$ ,  $\phi, \Phi, \gamma, \Gamma \in \mathbb{K}$  with  $\operatorname{Re}(\Phi \bar{\phi}) > 0$  and  $x, y \in H$  such that either

$$\operatorname{Re} \langle \Phi e - x, x - \phi e \rangle \geq 0, \quad \operatorname{Re} \langle \Gamma e - y, y - \gamma e \rangle \geq 0,$$

or equivalently,

$$\left\| x - \frac{\phi + \Phi}{2} e \right\| \leq \frac{1}{2} |\Phi - \phi|, \quad \left\| y - \frac{\gamma + \Gamma}{2} e \right\| \leq \frac{1}{2} |\Gamma - \gamma|$$

hold, then

$$0 \leq |\langle x, y \rangle - \langle x, e \rangle \langle e, y \rangle| \leq \frac{1}{4} M(\Phi, \phi) M(\Gamma, \gamma) |\langle x, e \rangle \langle e, y \rangle|,$$

where

$$M(\Phi, \phi) := \left[ \frac{(|\Phi| - |\phi|)^2 + 4 [|\phi \Phi| - \operatorname{Re}(\Phi \bar{\phi})]}{\operatorname{Re}(\Phi \bar{\phi})} \right]^{\frac{1}{2}}.$$

The constant  $\frac{1}{4}$  is best possible.

**Remark 144** If  $H$  is real,  $e \in H$ ,  $\|e\| = 1$  and  $a, b, A, B \in \mathbb{R}$  are such that  $A > a > 0$ ,  $B > b > 0$  and

$$\left\| x - \frac{a + A}{2} e \right\| \leq \frac{1}{2} (A - a), \quad \left\| y - \frac{b + B}{2} e \right\| \leq \frac{1}{2} (B - b),$$

then

$$|\langle x, y \rangle - \langle x, e \rangle \langle e, y \rangle| \leq \frac{1}{4} \cdot \frac{(A - a)(B - b)}{\sqrt{abAB}} |\langle x, e \rangle \langle e, y \rangle|. \quad (3.64)$$

The constant  $\frac{1}{4}$  is best possible.

If  $\langle x, e \rangle, \langle y, e \rangle \neq 0$ , then the following equivalent form of (3.64) also holds

$$\left| \frac{\langle x, y \rangle}{\langle x, e \rangle \langle e, y \rangle} - 1 \right| \leq \frac{1}{4} \cdot \frac{(A-a)(B-b)}{\sqrt{abAB}}.$$

### 3.4.3 Some Companion Inequalities

The following companion of the Grüss inequality also holds [3].

**Theorem 145** Let  $\{e_i\}_{i \in I}$  be a family of orthonormal vectors in  $H$ ,  $F$  a finite part of  $I$ ,  $\phi_i, \Phi_i \in \mathbb{K}$ , ( $i \in F$ ),  $x, y \in H$  and  $\lambda \in (0, 1)$ , such that either

$$\operatorname{Re} \left\langle \sum_{i \in F} \Phi_i e_i - (\lambda x + (1 - \lambda) y), \lambda x + (1 - \lambda) y - \sum_{i \in F} \phi_i e_i \right\rangle \geq 0, \quad (3.65)$$

or equivalently,

$$\left\| \lambda x + (1 - \lambda) y - \sum_{i \in F} \frac{\Phi_i + \phi_i}{2} \cdot e_i \right\| \leq \frac{1}{2} \left( \sum_{i \in F} |\Phi_i - \phi_i|^2 \right)^{\frac{1}{2}},$$

holds. Then we have the inequality

$$\begin{aligned} & \operatorname{Re} \left[ \langle x, y \rangle - \sum_{i \in F} \langle x, e_i \rangle \langle e_i, y \rangle \right] \\ & \leq \frac{1}{16} \cdot \frac{1}{\lambda(1 - \lambda)} \sum_{i \in F} M^2(\Phi, \phi, F) \sum_{i \in F} |\langle \lambda x + (1 - \lambda) y, e_i \rangle|^2. \end{aligned} \quad (3.66)$$

The constant  $\frac{1}{16}$  is the best possible constant in (3.66) in the sense that it cannot be replaced by a smaller constant.

**Proof.** Using the known inequality

$$\operatorname{Re} \langle z, u \rangle \leq \frac{1}{4} \|z + u\|^2$$

we may state that for any  $a, b \in H$  and  $\lambda \in (0, 1)$

$$\operatorname{Re} \langle a, b \rangle \leq \frac{1}{4\lambda(1 - \lambda)} \|\lambda a + (1 - \lambda) b\|^2. \quad (3.67)$$

Since

$$\langle x, y \rangle - \sum_{i \in F} \langle x, e_i \rangle \langle e_i, y \rangle = \left\langle x - \sum_{i \in F} \langle x, e_i \rangle e_i, y - \sum_{i \in F} \langle y, e_i \rangle e_i \right\rangle,$$

for any  $x, y \in H$ , then, by (3.67), we get

$$\begin{aligned}
& \operatorname{Re} \left[ \langle x, y \rangle - \sum_{i \in F} \langle x, e_i \rangle \langle e_i, y \rangle \right] \\
&= \operatorname{Re} \left[ \left\langle x - \sum_{i \in F} \langle x, e_i \rangle e_i, y - \sum_{i \in F} \langle y, e_i \rangle e_i \right\rangle \right] \\
&\leq \frac{1}{4\lambda(1-\lambda)} \left\| \lambda \left( x - \sum_{i \in F} \langle x, e_i \rangle e_i \right) + (1-\lambda) \left( y - \sum_{i \in F} \langle y, e_i \rangle e_i \right) \right\|^2 \\
&= \frac{1}{4\lambda(1-\lambda)} \left\| \lambda x + (1-\lambda)y - \sum_{i \in F} \langle \lambda x + (1-\lambda)y, e_i \rangle e_i \right\|^2 \\
&= \frac{1}{4\lambda(1-\lambda)} \left[ \|\lambda x + (1-\lambda)y\|^2 - \sum_{i \in F} |\langle \lambda x + (1-\lambda)y, e_i \rangle|^2 \right].
\end{aligned} \tag{3.68}$$

If we apply the reverse of Bessel's inequality from Corollary 138 for  $\lambda x + (1-\lambda)y$ , we may state that

$$\begin{aligned}
& \|\lambda x + (1-\lambda)y\|^2 - \sum_{i \in F} |\langle \lambda x + (1-\lambda)y, e_i \rangle|^2 \\
&\leq \frac{1}{4} M^2(\Phi, \phi, F) \sum_{i \in F} |\langle \lambda x + (1-\lambda)y, e_i \rangle|^2. \tag{3.69}
\end{aligned}$$

Now, by making use of (3.68) and (3.69), we deduce (3.66).

The fact that  $\frac{1}{16}$  is the best possible constant in (3.66) follows by the fact that if in (3.65) we choose  $x = y$ , then it becomes (i) of Theorem 136, implying for  $\lambda = \frac{1}{2}$  the inequality (3.52), for which, we have shown that  $\frac{1}{4}$  is the best constant. ■

**Remark 146** *If in Theorem 145, we choose  $\lambda = \frac{1}{2}$ , then we get*

$$\operatorname{Re} \left[ \langle x, y \rangle - \sum_{i \in F} \langle x, e_i \rangle \langle e_i, y \rangle \right] \leq \frac{1}{4} M^2(\Phi, \phi, F) \sum_{i \in F} \left| \left\langle \frac{x+y}{2}, e_i \right\rangle \right|^2,$$

*provided*

$$\operatorname{Re} \left\langle \sum_{i \in F} \Phi_i e_i - \frac{x+y}{2}, \frac{x+y}{2} - \sum_{i \in F} \phi_i e_i \right\rangle \geq 0$$

*or equivalently,*

$$\left\| \frac{x+y}{2} - \sum_{i \in F} \frac{\Phi_i + \phi_i}{2} \cdot e_i \right\| \leq \frac{1}{2} \left( \sum_{i \in F} |\Phi_i - \phi_i|^2 \right)^{\frac{1}{2}}.$$

### 3.4.4 Integral Inequalities

The following proposition holds [3].

**Proposition 147** *Let  $\{f_i\}_{i \in I}$  be an orthonormal family of functions in  $L^2_\rho(\Omega, \mathbb{K})$ ,  $F$  a finite subset of  $I$ ,  $\phi_i, \Phi_i \in \mathbb{K}$  ( $i \in F$ ) such that  $\sum_{i \in F} \operatorname{Re}(\Phi_i \overline{\phi_i}) > 0$  and  $f \in L^2_\rho(\Omega, \mathbb{K})$ , so that either*

$$\int_{\Omega} \rho(s) \operatorname{Re} \left[ \left( \sum_{i \in F} \Phi_i f_i(s) - f(s) \right) \left( \overline{f}(s) - \sum_{i \in F} \overline{\phi_i} \overline{f_i}(s) \right) \right] d\mu(s) \geq 0,$$

or equivalently,

$$\int_{\Omega} \rho(s) \left| f(s) - \sum_{i \in F} \frac{\Phi_i + \phi_i}{2} f_i(s) \right|^2 d\mu(s) \leq \frac{1}{4} \sum_{i \in F} |\Phi_i - \phi_i|^2.$$

Then we have the inequality

$$\left( \int_{\Omega} \rho(s) |f(s)|^2 d\mu(s) \right)^{\frac{1}{2}} \leq \frac{1}{2} \cdot \frac{1}{[\sum_{i \in F} \operatorname{Re}(\Phi_i \overline{\phi_i})]^{\frac{1}{2}}} \times \begin{cases} \max_{i \in F} \{|\Phi_i| + |\phi_i|\} \sum_{i \in F} \left| \int_{\Omega} \rho(s) f(s) \overline{f_i}(s) d\mu(s) \right| \\ \left[ \sum_{i \in F} (|\Phi_i| + |\phi_i|)^p \right]^{\frac{1}{p}} \left( \sum_{i \in F} \left| \int_{\Omega} \rho(s) f(s) \overline{f_i}(s) d\mu(s) \right|^q \right)^{\frac{1}{q}}, \\ \text{for } p > 1, \frac{1}{p} + \frac{1}{q} = 1 \\ \max_{i \in F} \left| \int_{\Omega} \rho(s) f(s) \overline{f_i}(s) d\mu(s) \right| \sum_{i \in F} [|\Phi_i| + |\phi_i|]. \end{cases}$$

In particular, we have

$$\begin{aligned} & \int_{\Omega} \rho(s) |f(s)|^2 d\mu(s) \\ & \leq \frac{1}{4} \cdot \frac{\sum_{i \in F} (|\Phi_i| + |\phi_i|)^2}{\sum_{i \in F} \operatorname{Re}(\Phi_i \overline{\phi_i})} \sum_{i \in F} \left| \int_{\Omega} \rho(s) f(s) \overline{f_i}(s) d\mu(s) \right|^2. \end{aligned} \quad (3.70)$$

The constant  $\frac{1}{4}$  is best possible.

The proof is obvious by Theorem 136 and Remark 137. We omit the details. The following proposition also holds.

**Proposition 148** Assume that  $f_i, f, \phi_i, \Phi_i$  and  $F$  satisfy the assumptions of Proposition 147. Then we have the following reverse of Bessel's inequality:

$$\begin{aligned} 0 &\leq \int_{\Omega} \rho(s) f^2(s) d\mu(s) - \sum_{i \in F} \left| \int_{\Omega} \rho(s) f(s) \overline{f_i}(s) d\mu(s) \right|^2 \\ &\leq \frac{1}{4} M^2(\Phi, \phi, F) \cdot \sum_{i \in F} \left| \int_{\Omega} \rho(s) f(s) \overline{f_i}(s) d\mu(s) \right|^2, \end{aligned} \quad (3.71)$$

where, as above,

$$M(\Phi, \phi, F) := \left[ \frac{\sum_{i \in F} \left\{ (|\Phi_i| - |\phi_i|)^2 + 4 [|\phi_i \Phi_i| - \operatorname{Re}(\Phi_i \overline{\phi_i})] \right\}}{\operatorname{Re}(\Phi_i \overline{\phi_i})} \right]^{\frac{1}{2}}. \quad (3.72)$$

The constant  $\frac{1}{4}$  is the best possible.

The following Grüss type inequality also holds.

**Proposition 149** Let  $\{f_i\}_{i \in I}$  and  $F$  be as in Proposition 147. If  $\phi_i, \Phi_i, \gamma_i, \Gamma_i \in \mathbb{K}$  ( $i \in F$ ) and  $f, g \in L^2_{\rho}(\Omega, \mathbb{K})$  such that either

$$\begin{aligned} \int_{\Omega} \rho(s) \operatorname{Re} \left[ \left( \sum_{i \in F} \Phi_i f_i(s) - f(s) \right) \left( \overline{f}(s) - \sum_{i \in F} \overline{\phi_i} \overline{f_i}(s) \right) \right] d\mu(s) &\geq 0, \\ \int_{\Omega} \rho(s) \operatorname{Re} \left[ \left( \sum_{i \in F} \Gamma_i f_i(s) - g(s) \right) \left( \overline{g}(s) - \sum_{i \in F} \overline{\gamma_i} \overline{f_i}(s) \right) \right] d\mu(s) &\geq 0, \end{aligned}$$

or equivalently,

$$\begin{aligned} \int_{\Omega} \rho(s) \left| f(s) - \sum_{i \in F} \frac{\Phi_i + \phi_i}{2} \cdot f_i(s) \right|^2 d\mu(s) &\leq \frac{1}{4} \sum_{i \in F} |\Phi_i - \phi_i|^2, \\ \int_{\Omega} \rho(s) \left| g(s) - \sum_{i \in F} \frac{\Gamma_i + \gamma_i}{2} \cdot f_i(s) \right|^2 d\mu(s) &\leq \frac{1}{4} \sum_{i \in F} |\Gamma_i - \gamma_i|^2, \end{aligned}$$

hold, then we have the inequality

$$\begin{aligned}
& \left| \int_{\Omega} \rho(s) f(s) \overline{g(s)} d\mu(s) \right. \\
& \quad \left. - \sum_{i \in F} \int_{\Omega} \rho(s) f(s) \overline{f_i(s)} d\mu(s) \int_{\Omega} \rho(s) f_i(s) \overline{g(s)} d\mu(s) \right| \\
& \leq \frac{1}{4} M(\Phi, \phi, F) M(\Gamma, \gamma, F) \left( \sum_{i \in F} \left| \int_{\Omega} \rho(s) f(s) \overline{f_i(s)} d\mu(s) \right|^2 \right)^{\frac{1}{2}} \\
& \quad \times \left( \sum_{i \in F} \left| \int_{\Omega} \rho(s) f_i(s) \overline{g(s)} d\mu(s) \right|^2 \right)^{\frac{1}{2}}, \quad (3.73)
\end{aligned}$$

where  $M(\Phi, \phi, F)$  is as defined in (3.72).

The constant  $\frac{1}{4}$  is the best possible.

The proof follows by Theorem 141 and we omit the details.

In the case of real spaces, the following corollaries provide much simpler sufficient conditions for the reverse of Bessel's inequality (3.71) or for the Grüss type inequality (3.73) to hold.

**Corollary 150** *Let  $\{f_i\}_{i \in I}$  be an orthornormal family of functions in the real Hilbert space  $L^2_{\rho}(\Omega)$ ,  $F$  a finite part of  $I$ ,  $M_i, m_i \geq 0$  ( $i \in F$ ), with  $\sum_{i \in F} M_i m_i > 0$  and  $f \in L^2_{\rho}(\Omega)$  so that*

$$\sum_{i \in F} m_i f_i(s) \leq f(s) \leq \sum_{i \in F} M_i f_i(s) \quad \text{for } \mu - a.e. \ s \in \Omega.$$

Then we have the inequality

$$\begin{aligned}
0 & \leq \int_{\Omega} \rho(s) f^2(s) d\mu(s) - \sum_{i \in F} \left[ \int_{\Omega} \rho(s) f(s) f_i(s) d\mu(s) \right]^2 \\
& \leq \frac{1}{4} \cdot \frac{\sum_{i \in F} (M_i - m_i)^2}{\sum_{i \in F} M_i m_i} \cdot \sum_{i \in F} \left[ \int_{\Omega} \rho(s) f(s) f_i(s) d\mu(s) \right]^2.
\end{aligned}$$

The constant  $\frac{1}{4}$  is best possible.

**Corollary 151** *Let  $\{f_i\}_{i \in I}$  and  $F$  be as above. If  $M_i, m_i, N_i, n_i \geq 0$  ( $i \in F$ ) with  $\sum_{i \in F} M_i m_i, \sum_{i \in F} N_i n_i > 0$  and  $f, g \in L^2_{\rho}(\Omega)$  such that*

$$\sum_{i \in F} m_i f_i(s) \leq f(s) \leq \sum_{i \in F} M_i f_i(s)$$

and

$$\sum_{i \in F} n_i f_i(s) \leq g(s) \leq \sum_{i \in F} N_i f_i(s) \quad \text{for } \mu - a.e. \ s \in \Omega,$$



then we have the inequality

$$\begin{aligned}
& \left| \int_{\Omega} \rho(s) f(s) g(s) d\mu(s) \right. \\
& \quad \left. - \sum_{i \in F} \int_{\Omega} \rho(s) f(s) f_i(s) d\mu(s) \int_{\Omega} \rho(s) g(s) f_i(s) d\mu(s) \right| \\
& \leq \frac{1}{4} \left( \frac{\sum_{i \in F} (M_i - m_i)^2}{\sum_{i \in F} M_i m_i} \right)^{\frac{1}{2}} \left( \frac{\sum_{i \in F} (N_i - n_i)^2}{\sum_{i \in F} N_i n_i} \right)^{\frac{1}{2}} \\
& \quad \times \left[ \sum_{i \in F} \left( \int_{\Omega} \rho(s) f(s) f_i(s) d\mu(s) \right)^2 \sum_{i \in F} \left( \int_{\Omega} \rho(s) g(s) f_i(s) d\mu(s) \right)^2 \right]^{\frac{1}{2}}.
\end{aligned}$$

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## 3.5 General Reverses of Bessel's Inequality in Hilbert Spaces

### 3.5.1 Some Reverses of Bessel's Inequality

Let  $(H; \langle \cdot, \cdot \rangle)$  be a real or complex infinite dimensional Hilbert space and  $(e_i)_{i \in \mathbb{N}}$  an orthonormal family in  $H$ , i.e., we recall that  $\langle e_i, e_j \rangle = 0$  if  $i, j \in \mathbb{N}$ ,  $i \neq j$  and  $\|e_i\| = 1$  for  $i \in \mathbb{N}$ .

It is well known that, if  $x \in H$ , then the sum  $\sum_{i=1}^{\infty} |\langle x, e_i \rangle|^2$  is convergent and the following inequality, called *Bessel's inequality*

$$\sum_{i=1}^{\infty} |\langle x, e_i \rangle|^2 \leq \|x\|^2, \quad (3.74)$$

holds.

If  $\ell^2(\mathbb{K}) := \left\{ \mathbf{a} = (a_i)_{i \in \mathbb{N}} \subset \mathbb{K} \mid \sum_{i=1}^{\infty} |a_i|^2 < \infty \right\}$ , where  $\mathbb{K} = \mathbb{C}$  or  $\mathbb{K} = \mathbb{R}$ , is the Hilbert space of all complex or real sequences that are 2-summable and  $\boldsymbol{\lambda} = (\lambda_i)_{i \in \mathbb{N}} \in \ell^2(\mathbb{K})$ , then the sum  $\sum_{i=1}^{\infty} \lambda_i e_i$  is convergent in  $H$  and if  $y := \sum_{i=1}^{\infty} \lambda_i e_i \in H$ , then  $\|y\| = \left( \sum_{i=1}^{\infty} |\lambda_i|^2 \right)^{\frac{1}{2}}$ .

We may state the following result [6].

**Theorem 152** *Let  $(H; \langle \cdot, \cdot \rangle)$  be an infinite dimensional Hilbert space over the real or complex number field  $\mathbb{K}$ ,  $(e_i)_{i \in \mathbb{N}}$  an orthonormal family in  $H$ ,  $\boldsymbol{\lambda} = (\lambda_i)_{i \in \mathbb{N}} \in \ell^2(\mathbb{K})$  and  $r > 0$  with the property that*

$$\sum_{i=1}^{\infty} |\lambda_i|^2 > r^2.$$

*If  $x \in H$  is such that*

$$\left\| x - \sum_{i=1}^{\infty} \lambda_i e_i \right\| \leq r,$$

*then we have the inequality*

$$\begin{aligned} \|x\|^2 &\leq \frac{\left( \sum_{i=1}^{\infty} \operatorname{Re} [\overline{\lambda_i} \langle x, e_i \rangle] \right)^2}{\sum_{i=1}^{\infty} |\lambda_i|^2 - r^2} \\ &\leq \frac{\left| \sum_{i=1}^{\infty} \overline{\lambda_i} \langle x, e_i \rangle \right|^2}{\sum_{i=1}^{\infty} |\lambda_i|^2 - r^2} \\ &\leq \frac{\sum_{i=1}^{\infty} |\lambda_i|^2}{\sum_{i=1}^{\infty} |\lambda_i|^2 - r^2} \sum_{i=1}^{\infty} |\langle x, e_i \rangle|^2 \end{aligned} \quad (3.75)$$

and

$$\begin{aligned} 0 &\leq \|x\|^2 - \sum_{i=1}^{\infty} |\langle x, e_i \rangle|^2 \\ &\leq \frac{r^2}{\sum_{i=1}^{\infty} |\lambda_i|^2 - r^2} \sum_{i=1}^{\infty} |\langle x, e_i \rangle|^2. \end{aligned} \quad (3.76)$$

**Proof.** Applying the third inequality in (1.48) for  $a = \sum_{i=1}^{\infty} \lambda_i e_i \in H$ , we have

$$\|x\|^2 \left\| \sum_{i=1}^{\infty} \lambda_i e_i \right\|^2 - \left[ \operatorname{Re} \left\langle x, \sum_{i=1}^{\infty} \lambda_i e_i \right\rangle \right]^2 \leq r^2 \|x\|^2 \quad (3.77)$$

and since

$$\begin{aligned} \left\| \sum_{i=1}^{\infty} \lambda_i e_i \right\|^2 &= \sum_{i=1}^{\infty} |\lambda_i|^2, \\ \operatorname{Re} \left\langle x, \sum_{i=1}^{\infty} \lambda_i e_i \right\rangle &= \sum_{i=1}^{\infty} \operatorname{Re} [\overline{\lambda_i} \langle x, e_i \rangle], \end{aligned}$$

then, by (3.77), we deduce

$$\|x\|^2 \sum_{i=1}^{\infty} |\lambda_i|^2 - \left[ \operatorname{Re} \left\langle x, \sum_{i=1}^{\infty} \lambda_i e_i \right\rangle \right]^2 \leq r^2 \|x\|^2,$$

giving the first inequality in (3.75).

The second inequality is obvious by the modulus property.

The last inequality follows by the Cauchy-Bunyakovsky-Schwarz inequality

$$\left| \sum_{i=1}^{\infty} \overline{\lambda_i} \langle x, e_i \rangle \right|^2 \leq \sum_{i=1}^{\infty} |\lambda_i|^2 \sum_{i=1}^{\infty} |\langle x, e_i \rangle|^2.$$

The inequality (3.76) follows by the last inequality in (3.75) on subtracting from both sides the quantity  $\sum_{i=1}^{\infty} |\langle x, e_i \rangle|^2 < \infty$ . ■

The following result provides a generalization for the reverse of Bessel's inequality obtained in [5].

**Theorem 153** *Let  $(H; \langle \cdot, \cdot \rangle)$  and  $(e_i)_{i \in \mathbb{N}}$  be as in Theorem 152. Suppose that  $\Gamma = (\Gamma_i)_{i \in \mathbb{N}} \in \ell^2(\mathbb{K})$ ,  $\gamma = (\gamma_i)_{i \in \mathbb{N}} \in \ell^2(\mathbb{K})$  are sequences of real or complex numbers such that*

$$\sum_{i=1}^{\infty} \operatorname{Re}(\Gamma_i \overline{\gamma_i}) > 0.$$

*If  $x \in H$  is such that either*

$$\left\| x - \sum_{i=1}^{\infty} \frac{\Gamma_i + \gamma_i}{2} e_i \right\| \leq \frac{1}{2} \left( \sum_{i=1}^{\infty} |\Gamma_i - \gamma_i|^2 \right)^{\frac{1}{2}} \quad (3.78)$$

or equivalently,

$$\operatorname{Re} \left\langle \sum_{i=1}^{\infty} \Gamma_i e_i - x, x - \sum_{i=1}^{\infty} \gamma_i e_i \right\rangle \geq 0 \quad (3.79)$$

holds, then we have the inequalities

$$\begin{aligned} \|x\|^2 &\leq \frac{1}{4} \cdot \frac{\left( \sum_{i=1}^{\infty} \operatorname{Re} [(\overline{\Gamma_i} + \overline{\gamma_i}) \langle x, e_i \rangle] \right)^2}{\sum_{i=1}^{\infty} \operatorname{Re} (\Gamma_i \overline{\gamma_i})} \\ &\leq \frac{1}{4} \cdot \frac{\left| \sum_{i=1}^{\infty} (\overline{\Gamma_i} + \overline{\gamma_i}) \langle x, e_i \rangle \right|^2}{\sum_{i=1}^{\infty} \operatorname{Re} (\Gamma_i \overline{\gamma_i})} \\ &\leq \frac{1}{4} \cdot \frac{\sum_{i=1}^{\infty} |\Gamma_i + \gamma_i|^2}{\sum_{i=1}^{\infty} \operatorname{Re} (\Gamma_i \overline{\gamma_i})} \sum_{i=1}^{\infty} |\langle x, e_i \rangle|^2. \end{aligned} \quad (3.80)$$

The constant  $\frac{1}{4}$  is best possible in all inequalities in (3.80).

We also have the inequalities:

$$0 \leq \|x\|^2 - \sum_{i=1}^{\infty} |\langle x, e_i \rangle|^2 \leq \frac{1}{4} \cdot \frac{\sum_{i=1}^{\infty} |\Gamma_i - \gamma_i|^2}{\sum_{i=1}^{\infty} \operatorname{Re} (\Gamma_i \overline{\gamma_i})} \sum_{i=1}^{\infty} |\langle x, e_i \rangle|^2. \quad (3.81)$$

Here the constant  $\frac{1}{4}$  is also best possible.

**Proof.** Since  $\Gamma, \gamma \in \ell^2(\mathbb{K})$ , then also  $\frac{1}{2}(\Gamma \pm \gamma) \in \ell^2(\mathbb{K})$ , showing that the series

$$\sum_{i=1}^{\infty} \left| \frac{\Gamma_i + \gamma_i}{2} \right|^2, \quad \sum_{i=1}^{\infty} \left| \frac{\Gamma_i - \gamma_i}{2} \right|^2 \quad \text{and} \quad \sum_{i=1}^{\infty} \operatorname{Re} (\Gamma_i \overline{\gamma_i})$$

are convergent. Also, the series

$$\sum_{i=1}^{\infty} \Gamma_i e_i, \quad \sum_{i=1}^{\infty} \gamma_i e_i \quad \text{and} \quad \sum_{i=1}^{\infty} \frac{\gamma_i + \Gamma_i}{2} e_i$$

are convergent in the Hilbert space  $H$ .

Now, we observe that the inequalities (3.80) and (3.81) follow from Theorem 152 on choosing  $\lambda_i = \frac{\gamma_i + \Gamma_i}{2}$ ,  $i \in \mathbb{N}$  and  $r = \frac{1}{2} \left( \sum_{i=1}^{\infty} |\Gamma_i - \gamma_i|^2 \right)^{\frac{1}{2}}$ .

The fact that  $\frac{1}{4}$  is the best constant in both (3.80) and (3.81) follows from Theorem 38 and Corollary 40, and we omit the details. ■

For some recent results related to the Bessel inequality, see [1], [2], [8], and [9].

### 3.5.2 Some Grüss Type Inequalities for Orthonormal Families

The following result related to the Grüss inequality in inner product spaces holds [6].

**Theorem 154** Let  $(H; \langle \cdot, \cdot \rangle)$  be an infinite dimensional Hilbert space over the real or complex number field  $\mathbb{K}$ , and  $(e_i)_{i \in \mathbb{N}}$  an orthornormal family in  $H$ . Assume that  $\lambda = (\lambda_i)_{i \in \mathbb{N}}, \mu = (\mu_i)_{i \in \mathbb{N}} \in \ell^2(\mathbb{K})$  and  $r_1, r_2 > 0$  with the properties that

$$\sum_{i=1}^{\infty} |\lambda_i|^2 > r_1^2, \quad \sum_{i=1}^{\infty} |\mu_i|^2 > r_2^2.$$

If  $x, y \in H$  are such that

$$\left\| x - \sum_{i=1}^{\infty} \lambda_i e_i \right\| \leq r_1, \quad \left\| y - \sum_{i=1}^{\infty} \mu_i e_i \right\| \leq r_2,$$

then we have the inequalities

$$\begin{aligned} & \left| \langle x, y \rangle - \sum_{i=1}^{\infty} \langle x, e_i \rangle \langle e_i, y \rangle \right| \\ & \leq \frac{r_1 r_2}{\sqrt{\sum_{i=1}^{\infty} |\lambda_i|^2 - r_1^2} \sqrt{\sum_{i=1}^{\infty} |\mu_i|^2 - r_2^2}} \cdot \sqrt{\sum_{i=1}^{\infty} |\langle x, e_i \rangle|^2 \sum_{i=1}^{\infty} |\langle y, e_i \rangle|^2} \\ & \leq \frac{r_1 r_2 \|x\| \|y\|}{\sqrt{\sum_{i=1}^{\infty} |\lambda_i|^2 - r_1^2} \sqrt{\sum_{i=1}^{\infty} |\mu_i|^2 - r_2^2}}. \end{aligned} \quad (3.82)$$

**Proof.** Applying Schwarz's inequality for the vectors  $x - \sum_{i=1}^{\infty} \langle x, e_i \rangle e_i$ ,  $y - \sum_{i=1}^{\infty} \langle y, e_i \rangle e_i$ , we have

$$\begin{aligned} & \left| \left\langle x - \sum_{i=1}^{\infty} \langle x, e_i \rangle e_i, y - \sum_{i=1}^{\infty} \langle y, e_i \rangle e_i \right\rangle \right|^2 \\ & \leq \left\| x - \sum_{i=1}^{\infty} \langle x, e_i \rangle e_i \right\|^2 \left\| y - \sum_{i=1}^{\infty} \langle y, e_i \rangle e_i \right\|^2. \end{aligned} \quad (3.83)$$

Since

$$\left\langle x - \sum_{i=1}^{\infty} \langle x, e_i \rangle e_i, y - \sum_{i=1}^{\infty} \langle y, e_i \rangle e_i \right\rangle = \langle x, y \rangle - \sum_{i=1}^{\infty} \langle x, e_i \rangle \langle e_i, y \rangle$$

and

$$\left\| x - \sum_{i=1}^{\infty} \langle x, e_i \rangle e_i \right\|^2 = \|x\|^2 - \sum_{i=1}^{\infty} |\langle x, e_i \rangle|^2,$$

then by (3.76) applied for  $x$  and  $y$ , and from (3.83), we deduce the first part of (3.82).

The second part follows by Bessel's inequality. ■

The following Grüss type inequality may be stated as well.

**Theorem 155** Let  $(H; \langle \cdot, \cdot \rangle)$  be an infinite dimensional Hilbert space and  $(e_i)_{i \in \mathbb{N}}$  an orthonormal family in  $H$ . Suppose that  $(\Gamma_i)_{i \in \mathbb{N}}, (\gamma_i)_{i \in \mathbb{N}}, (\phi_i)_{i \in \mathbb{N}}, (\Phi_i)_{i \in \mathbb{N}} \in \ell^2(\mathbb{K})$  are sequences of real and complex numbers such that

$$\sum_{i=1}^{\infty} \operatorname{Re}(\Gamma_i \overline{\gamma_i}) > 0, \quad \sum_{i=1}^{\infty} \operatorname{Re}(\Phi_i \overline{\phi_i}) > 0.$$

If  $x, y \in H$  are such that either

$$\left\| x - \sum_{i=1}^{\infty} \frac{\Gamma_i + \gamma_i}{2} \cdot e_i \right\| \leq \frac{1}{2} \left( \sum_{i=1}^{\infty} |\Gamma_i - \gamma_i|^2 \right)^{\frac{1}{2}},$$

$$\left\| y - \sum_{i=1}^{\infty} \frac{\Phi_i + \phi_i}{2} \cdot e_i \right\| \leq \frac{1}{2} \left( \sum_{i=1}^{\infty} |\Phi_i - \phi_i|^2 \right)^{\frac{1}{2}},$$

or equivalently,

$$\operatorname{Re} \left\langle \sum_{i=1}^{\infty} \Gamma_i e_i - x, x - \sum_{i=1}^{\infty} \gamma_i e_i \right\rangle \geq 0,$$

$$\operatorname{Re} \left\langle \sum_{i=1}^{\infty} \Phi_i e_i - y, y - \sum_{i=1}^{\infty} \phi_i e_i \right\rangle \geq 0,$$

holds, then we have the inequality

$$\begin{aligned} & \left| \langle x, y \rangle - \sum_{i=1}^{\infty} \langle x, e_i \rangle \langle e_i, y \rangle \right| \\ & \leq \frac{1}{4} \cdot \frac{\left( \sum_{i=1}^{\infty} |\Gamma_i - \gamma_i|^2 \right)^{\frac{1}{2}} \left( \sum_{i=1}^{\infty} |\Phi_i - \phi_i|^2 \right)^{\frac{1}{2}}}{\left( \sum_{i=1}^{\infty} \operatorname{Re}(\Gamma_i \overline{\gamma_i}) \right)^{\frac{1}{2}} \left( \sum_{i=1}^{\infty} \operatorname{Re}(\Phi_i \overline{\phi_i}) \right)^{\frac{1}{2}}} \\ & \quad \times \left( \sum_{i=1}^{\infty} |\langle x, e_i \rangle|^2 \right)^{\frac{1}{2}} \left( \sum_{i=1}^{\infty} |\langle y, e_i \rangle|^2 \right)^{\frac{1}{2}} \\ & \leq \frac{1}{4} \cdot \frac{\left( \sum_{i=1}^{\infty} |\Gamma_i - \gamma_i|^2 \right)^{\frac{1}{2}} \left( \sum_{i=1}^{\infty} |\Phi_i - \phi_i|^2 \right)^{\frac{1}{2}}}{\left[ \sum_{i=1}^{\infty} \operatorname{Re}(\Gamma_i \overline{\gamma_i}) \right]^{\frac{1}{2}} \left[ \sum_{i=1}^{\infty} \operatorname{Re}(\Phi_i \overline{\phi_i}) \right]^{\frac{1}{2}}} \|x\| \|y\| \end{aligned}$$

The constant  $\frac{1}{4}$  is best possible in the first inequality.

**Proof.** Follows by (3.81) and (3.83). The best constant follows from Theorem 97, and we omit the details. ■

### 3.5.3 Other Reverses of Bessel's Inequality

We may state the following result [7].

**Theorem 156** Let  $(H; \langle \cdot, \cdot \rangle)$  be an infinite dimensional Hilbert space over the real or complex number field  $\mathbb{K}$ ,  $(e_i)_{i \in \mathbb{N}}$  is an orthonormal family in  $H$ ,  $\lambda = (\lambda_i)_{i \in \mathbb{N}} \in \ell^2(\mathbb{K})$ ,  $\lambda \neq 0$  and  $r > 0$ . If  $x \in H$  is such that

$$\left\| x - \sum_{i=1}^{\infty} \lambda_i e_i \right\| \leq r,$$

then we have the inequality

$$0 \leq \|x\| - \left( \sum_{i=1}^{\infty} |\langle x, e_i \rangle|^2 \right)^{\frac{1}{2}} \leq \frac{1}{2} \cdot \frac{r^2}{\left( \sum_{i=1}^{\infty} |\lambda_i|^2 \right)^{\frac{1}{2}}}. \quad (3.84)$$

The constant  $\frac{1}{2}$  is best possible in (3.84) in the sense that it cannot be replaced by a smaller constant.

**Proof.** Let  $a := \sum_{i=1}^{\infty} \lambda_i e_i \in H$ . Then by Theorem 47, we have

$$\|x\| \left\| \sum_{i=1}^{\infty} \lambda_i e_i \right\| - \left| \sum_{i=1}^{\infty} \bar{\lambda}_i \langle x, e_i \rangle \right| \leq \frac{1}{2} r^2,$$

giving

$$\|x\| \left( \sum_{i=1}^{\infty} |\lambda_i|^2 \right)^{\frac{1}{2}} \leq \frac{1}{2} r^2 + \left| \sum_{i=1}^{\infty} \bar{\lambda}_i \langle x, e_i \rangle \right|, \quad (3.85)$$

since

$$\left\| \sum_{i=1}^{\infty} \lambda_i e_i \right\| = \left( \sum_{i=1}^{\infty} |\lambda_i|^2 \right)^{\frac{1}{2}}.$$

Using the Cauchy-Bunyakovsky-Schwarz inequality, we may state that

$$\left| \sum_{i=1}^{\infty} \bar{\lambda}_i \langle x, e_i \rangle \right| \leq \left( \sum_{i=1}^{\infty} |\lambda_i|^2 \right)^{\frac{1}{2}} \left( \sum_{i=1}^{\infty} |\langle x, e_i \rangle|^2 \right)^{\frac{1}{2}}, \quad (3.86)$$

and thus, by (3.85) and (3.86), we may state that

$$\|x\| \left( \sum_{i=1}^{\infty} |\lambda_i|^2 \right)^{\frac{1}{2}} \leq \frac{1}{2} r^2 + \left( \sum_{i=1}^{\infty} |\lambda_i|^2 \right)^{\frac{1}{2}} \left( \sum_{i=1}^{\infty} |\langle x, e_i \rangle|^2 \right)^{\frac{1}{2}},$$

from where we get the desired inequality in (3.84).

The best constant, follows by Theorem 47 on choosing  $(e_i)_{i \in \mathbb{N}} = \{e\}$ , with  $\|e\| = 1$  and we omit the details. ■



**Remark 157** Under the assumptions of Theorem 156, and if we multiply by  $\|x\| + \left(\sum_{i=1}^{\infty} |\langle x, e_i \rangle|^2\right)^{\frac{1}{2}} > 0$ , then we deduce, from (3.84), that

$$\begin{aligned} 0 &\leq \|x\|^2 - \sum_{i=1}^{\infty} |\langle x, e_i \rangle|^2 \\ &\leq \frac{1}{2} \cdot \frac{r^2 \left( \|x\| + \left(\sum_{i=1}^{\infty} |\langle x, e_i \rangle|^2\right)^{\frac{1}{2}} \right)}{\left(\sum_{i=1}^{\infty} |\lambda_i|^2\right)^{\frac{1}{2}}} \\ &\leq \frac{r^2 \|x\|}{\left(\sum_{i=1}^{\infty} |\lambda_i|^2\right)^{\frac{1}{2}}}, \end{aligned} \tag{3.87}$$

where for the last inequality, we have used Bessel's inequality

$$\left( \sum_{i=1}^{\infty} |\langle x, e_i \rangle|^2 \right)^{\frac{1}{2}} \leq \|x\|, \quad x \in H.$$

The following result also holds [7].

**Theorem 158** Assume that  $(H; \langle \cdot, \cdot \rangle)$  and  $(e_i)_{i \in \mathbb{N}}$  are as in Theorem 156. If  $\mathbf{\Gamma} = (\Gamma_i)_{i \in \mathbb{N}}$ ,  $\gamma = (\gamma_i)_{i \in \mathbb{N}} \in \ell^2(\mathbb{K})$ , with  $\mathbf{\Gamma} \neq -\gamma$ , and  $x \in H$  with the property that, either

$$\left\| x - \sum_{i=1}^{\infty} \frac{\Gamma_i + \gamma_i}{2} \cdot e_i \right\| \leq \frac{1}{2} \left( \sum_{i=1}^{\infty} |\Gamma_i - \gamma_i|^2 \right)^{\frac{1}{2}},$$

or equivalently,

$$\operatorname{Re} \left\langle \sum_{i=1}^{\infty} \Gamma_i e_i - x, x - \sum_{i=1}^{\infty} \gamma_i e_i \right\rangle \geq 0,$$

holds, then we have the inequality

$$0 \leq \|x\| - \left( \sum_{i=1}^{\infty} |\langle x, e_i \rangle|^2 \right)^{\frac{1}{2}} \leq \frac{1}{4} \cdot \frac{\sum_{i=1}^{\infty} |\Gamma_i - \gamma_i|^2}{\left(\sum_{i=1}^{\infty} |\Gamma_i + \gamma_i|^2\right)^{\frac{1}{2}}}. \tag{3.88}$$

The constant  $\frac{1}{4}$  is best possible in the sense that it cannot be replaced by a smaller constant.

**Proof.** Since  $\mathbf{\Gamma}, \gamma \in \ell^2(\mathbb{K})$ , then we have that  $\frac{1}{2}(\mathbf{\Gamma} \pm \gamma) \in \ell^2(\mathbb{K})$ , showing that the series

$$\sum_{i=1}^{\infty} \left| \frac{\Gamma_i + \gamma_i}{2} \right|^2, \quad \sum_{i=1}^{\infty} \left| \frac{\Gamma_i - \gamma_i}{2} \right|^2$$

are convergent. In addition, the series  $\sum_{i=1}^{\infty} \Gamma_i e_i$ ,  $\sum_{i=1}^{\infty} \gamma_i e_i$  and  $\sum_{i=1}^{\infty} \frac{\Gamma_i + \gamma_i}{2} e_i$  are also convergent in the Hilbert space  $H$ .

Now, we observe that the inequality (3.88) follows from Theorem 156 on choosing  $\lambda_i = \frac{\Gamma_i + \gamma_i}{2}$ ,  $i \in \mathbb{N}$  and  $r = \frac{1}{2} \left( \sum_{i=1}^{\infty} |\Gamma_i - \gamma_i|^2 \right)^{\frac{1}{2}}$ .

The fact that  $\frac{1}{4}$  is the best possible constant in (3.88) follows from Theorem 48, and we omit the details. ■

**Remark 159** *With the assumptions of Theorem 158, we have*

$$\begin{aligned} 0 &\leq \|x\|^2 - \sum_{i=1}^{\infty} |\langle x, e_i \rangle|^2 \\ &\leq \frac{1}{4} \cdot \frac{\sum_{i=1}^{\infty} |\Gamma_i - \gamma_i|^2}{\left( \sum_{i=1}^{\infty} |\Gamma_i + \gamma_i|^2 \right)^{\frac{1}{2}}} \left[ \|x\| + \left( \sum_{i=1}^{\infty} |\langle x, e_i \rangle|^2 \right)^{\frac{1}{2}} \right] \\ &\leq \frac{1}{2} \cdot \frac{\sum_{i=1}^{\infty} |\Gamma_i - \gamma_i|^2}{\left( \sum_{i=1}^{\infty} |\Gamma_i + \gamma_i|^2 \right)^{\frac{1}{2}}} \|x\|. \end{aligned} \quad (3.89)$$

### 3.5.4 More Grüss Type Inequalities for Orthonormal Families

The following result holds [7].

**Theorem 160** *Let  $(H; \langle \cdot, \cdot \rangle)$  be an infinite dimensional Hilbert space over the real or complex number field  $\mathbb{K}$  and  $(e_i)_{i \in \mathbb{N}}$  an orthonormal family in  $H$ . If  $\lambda = (\lambda_i)_{i \in \mathbb{N}}$ ,  $\mu = (\mu_i)_{i \in \mathbb{N}} \in \ell^2(\mathbb{K})$ ,  $\lambda, \mu \neq 0$ ,  $r_1, r_2 > 0$  and  $x, y \in H$  are such that*

$$\left\| x - \sum_{i=1}^{\infty} \lambda_i e_i \right\| \leq r_1, \quad \left\| y - \sum_{i=1}^{\infty} \mu_i e_i \right\| \leq r_2,$$

*then we have the inequality*

$$\begin{aligned} &\left| \langle x, y \rangle - \sum_{i=1}^{\infty} \langle x, e_i \rangle \langle e_i, y \rangle \right| \\ &\leq \frac{1}{2} r_1 r_2 \frac{\left[ \|x\| + \left( \sum_{i=1}^{\infty} |\langle x, e_i \rangle|^2 \right)^{\frac{1}{2}} \right]^{\frac{1}{2}} \left[ \|y\| + \left( \sum_{i=1}^{\infty} |\langle y, e_i \rangle|^2 \right)^{\frac{1}{2}} \right]^{\frac{1}{2}}}{\left( \sum_{i=1}^{\infty} |\lambda_i|^2 \right)^{\frac{1}{4}} \left( \sum_{i=1}^{\infty} |\mu_i|^2 \right)^{\frac{1}{4}}} \\ &\leq r_1 r_2 \frac{\|x\|^{\frac{1}{2}} \|y\|^{\frac{1}{2}}}{\left( \sum_{i=1}^{\infty} |\lambda_i|^2 \right)^{\frac{1}{4}} \left( \sum_{i=1}^{\infty} |\mu_i|^2 \right)^{\frac{1}{4}}}. \end{aligned}$$

**Proof.** It follows by (3.87) applied for  $x$  and  $y$ . We omit the details. ■

Finally we may state the following theorem [7].

**Theorem 161** Assume that  $(H; \langle \cdot, \cdot \rangle)$  and  $(e_i)_{i \in \mathbb{N}}$  are as in Theorem 160. If  $\Gamma = (\Gamma_i)_{i \in \mathbb{N}}$ ,  $\mathbf{\Gamma} = (\Gamma_i)_{i \in \mathbb{N}}$ ,  $\phi = (\phi_i)_{i \in \mathbb{N}}$ ,  $\Phi = (\Phi_i)_{i \in \mathbb{N}} \in \ell^2(\mathbb{K})$ , with  $\Gamma \neq -\gamma$ ,  $\Phi \neq -\phi$ , and  $x, y \in H$  are such that, either

$$\left\| x - \sum_{i=1}^{\infty} \frac{\Gamma_i + \gamma_i}{2} \cdot e_i \right\| \leq \frac{1}{2} \left( \sum_{i=1}^{\infty} |\Gamma_i - \gamma_i|^2 \right)^{\frac{1}{2}},$$

$$\left\| y - \sum_{i=1}^{\infty} \frac{\Phi_i + \phi_i}{2} \cdot e_i \right\| \leq \frac{1}{2} \left( \sum_{i=1}^{\infty} |\Phi_i - \phi_i|^2 \right)^{\frac{1}{2}},$$

or equivalently,

$$\operatorname{Re} \left\langle \sum_{i=1}^{\infty} \Gamma_i e_i - x, x - \sum_{i=1}^{\infty} \gamma_i e_i \right\rangle \geq 0,$$

$$\operatorname{Re} \left\langle \sum_{i=1}^{\infty} \Phi_i e_i - y, y - \sum_{i=1}^{\infty} \phi_i e_i \right\rangle \geq 0,$$

holds, then we have the inequality

$$\begin{aligned} & \left| \langle x, y \rangle - \sum_{i=1}^{\infty} \langle x, e_i \rangle \langle e_i, y \rangle \right| \\ & \leq \frac{1}{4} \cdot \left( \sum_{i=1}^{\infty} |\Phi_i - \phi_i|^2 \right)^{\frac{1}{2}} \left( \sum_{i=1}^{\infty} |\Gamma_i - \gamma_i|^2 \right)^{\frac{1}{2}} \\ & \quad \times \frac{\left[ \|x\| + \left( \sum_{i=1}^{\infty} |\langle x, e_i \rangle|^2 \right)^{\frac{1}{2}} \right]^{\frac{1}{2}} \left[ \|y\| + \left( \sum_{i=1}^{\infty} |\langle y, e_i \rangle|^2 \right)^{\frac{1}{2}} \right]^{\frac{1}{2}}}{\left( \sum_{i=1}^{\infty} |\Phi_i + \phi_i|^2 \right)^{\frac{1}{4}} \left( \sum_{i=1}^{\infty} |\Gamma_i + \gamma_i|^2 \right)^{\frac{1}{4}}} \\ & \leq \frac{1}{2} \cdot \frac{\left( \sum_{i=1}^{\infty} |\Phi_i - \phi_i|^2 \right)^{\frac{1}{2}} \left( \sum_{i=1}^{\infty} |\Gamma_i - \gamma_i|^2 \right)^{\frac{1}{2}}}{\left( \sum_{i=1}^{\infty} |\Phi_i + \phi_i|^2 \right)^{\frac{1}{4}} \left( \sum_{i=1}^{\infty} |\Gamma_i + \gamma_i|^2 \right)^{\frac{1}{4}}} \|x\|^{\frac{1}{2}} \|y\|^{\frac{1}{2}}. \end{aligned}$$

The proof follow by (3.89) applied for  $x$  and  $y$ . We omit the details.

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## Chapter 4

# Generalisations of Bessel's Inequality

### 4.1 Boas-Bellman Type Inequalities

#### 4.1.1 Introduction

Let  $(H; (\cdot, \cdot))$  be an inner product space over the real or complex number field  $\mathbb{K}$ . If  $(e_i)_{1 \leq i \leq n}$  are orthonormal vectors in the inner product space  $H$ , i.e.,  $(e_i, e_j) = \delta_{ij}$  for all  $i, j \in \{1, \dots, n\}$ , where  $\delta_{ij}$  is the Kronecker delta, then we have the following inequality is well known in the literature as Bessel's inequality (see for example [7, p. 391]):

$$\sum_{i=1}^n |(x, e_i)|^2 \leq \|x\|^2 \quad \text{for any } x \in H.$$

For other results related to Bessel's inequality, see [4] – [6] and Chapter XV in the book [7].

In 1941, R.P. Boas [2] and in 1944, independently, R. Bellman [1] proved the following generalisation of Bessel's inequality (see also [7, p. 392]).

**Theorem 162** *If  $x, y_1, \dots, y_n$  are elements of an inner product space  $(H; (\cdot, \cdot))$ , then the following inequality:*

$$\sum_{i=1}^n |(x, y_i)|^2 \leq \|x\|^2 \left[ \max_{1 \leq i \leq n} \|y_i\|^2 + \left( \sum_{1 \leq i \neq j \leq n} |(y_i, y_j)|^2 \right)^{\frac{1}{2}} \right], \quad (4.1)$$

*holds.*

A recent generalisation of the Boas-Bellman result was given in Mitrinović-Pečarić-Fink [7, p. 392] where they proved the following:

**Theorem 163** *If  $x, y_1, \dots, y_n$  are as in Theorem 162 and  $c_1, \dots, c_n \in \mathbb{K}$ , then one has the inequality:*

$$\left| \sum_{i=1}^n c_i(x, y_i) \right|^2 \leq \|x\|^2 \sum_{i=1}^n |c_i|^2 \left[ \max_{1 \leq i \leq n} \|y_i\|^2 + \left( \sum_{1 \leq i \neq j \leq n} |(y_i, y_j)|^2 \right)^{\frac{1}{2}} \right]. \quad (4.2)$$

They also noted that if in (4.2) one chooses  $c_i = \overline{(x, y_i)}$ , then this inequality becomes (4.1).

For other results related to the Boas-Bellman inequality, see [5].

In this section, by following [3], we point out some new results that may be related to both the Mitrinović-Pečarić-Fink and Boas-Bellman inequalities.

#### 4.1.2 Preliminary Results

We start with the following lemma which is also interesting in itself [3].

**Lemma 164** *Let  $z_1, \dots, z_n \in H$  and  $\alpha_1, \dots, \alpha_n \in \mathbb{K}$ . Then one has the inequality:*

$$\left\| \sum_{i=1}^n \alpha_i z_i \right\|^2 \leq \begin{cases} \max_{1 \leq i \leq n} |\alpha_i|^2 \sum_{i=1}^n \|z_i\|^2; \\ \left( \sum_{i=1}^n |\alpha_i|^{2\alpha} \right)^{\frac{1}{\alpha}} \left( \sum_{i=1}^n \|z_i\|^{2\beta} \right)^{\frac{1}{\beta}}, & \text{where } \alpha > 1, \\ & \frac{1}{\alpha} + \frac{1}{\beta} = 1; \\ \sum_{i=1}^n |\alpha_i|^2 \max_{1 \leq i \leq n} \|z_i\|^2, \end{cases}$$

$$+ \begin{cases} \max_{1 \leq i \neq j \leq n} \{|\alpha_i \alpha_j|\} \sum_{1 \leq i \neq j \leq n} |(z_i, z_j)|; \\ \left[ \left( \sum_{i=1}^n |\alpha_i|^\gamma \right)^2 - \left( \sum_{i=1}^n |\alpha_i|^{2\gamma} \right) \right]^{\frac{1}{\gamma}} \left( \sum_{1 \leq i \neq j \leq n} |(z_i, z_j)|^\delta \right)^{\frac{1}{\delta}}, \\ & \text{where } \gamma > 1, \quad \frac{1}{\gamma} + \frac{1}{\delta} = 1; \\ \left[ \left( \sum_{i=1}^n |\alpha_i| \right)^2 - \sum_{i=1}^n |\alpha_i|^2 \right] \max_{1 \leq i \neq j \leq n} |(z_i, z_j)|. \end{cases} \quad (4.3)$$

**Proof.** We observe that

$$\begin{aligned}
\left\| \sum_{i=1}^n \alpha_i z_i \right\|^2 &= \left( \sum_{i=1}^n \alpha_i z_i, \sum_{j=1}^n \alpha_j z_j \right) \\
&= \sum_{i=1}^n \sum_{j=1}^n \alpha_i \overline{\alpha_j} (z_i, z_j) = \left| \sum_{i=1}^n \sum_{j=1}^n \alpha_i \overline{\alpha_j} (z_i, z_j) \right| \\
&\leq \sum_{i=1}^n \sum_{j=1}^n |\alpha_i| |\overline{\alpha_j}| |(z_i, z_j)| \\
&= \sum_{i=1}^n |\alpha_i|^2 \|z_i\|^2 + \sum_{1 \leq i \neq j \leq n} |\alpha_i| |\alpha_j| |(z_i, z_j)|.
\end{aligned} \tag{4.4}$$

Using Hölder's inequality, we may write that

$$\begin{aligned}
&\sum_{i=1}^n |\alpha_i|^2 \|z_i\|^2 \\
&\leq \begin{cases} \max_{1 \leq i \leq n} |\alpha_i|^2 \sum_{i=1}^n \|z_i\|^2; \\ \left( \sum_{i=1}^n |\alpha_i|^{2\alpha} \right)^{\frac{1}{\alpha}} \left( \sum_{i=1}^n \|z_i\|^{2\beta} \right)^{\frac{1}{\beta}}, \quad \text{where } \alpha > 1, \frac{1}{\alpha} + \frac{1}{\beta} = 1; \\ \sum_{i=1}^n |\alpha_i|^2 \max_{1 \leq i \leq n} \|z_i\|^2. \end{cases} \tag{4.5}
\end{aligned}$$

By Hölder's inequality for double sums we also have

$$\begin{aligned}
&\sum_{1 \leq i \neq j \leq n} |\alpha_i| |\alpha_j| |(z_i, z_j)| \\
&\leq \begin{cases} \max_{1 \leq i \neq j \leq n} |\alpha_i \alpha_j| \sum_{1 \leq i \neq j \leq n} |(z_i, z_j)|; \\ \left( \sum_{1 \leq i \neq j \leq n} |\alpha_i|^\gamma |\alpha_j|^\gamma \right)^{\frac{1}{\gamma}} \left( \sum_{1 \leq i \neq j \leq n} |(z_i, z_j)|^\delta \right)^{\frac{1}{\delta}}, \\ \quad \text{where } \gamma > 1, \frac{1}{\gamma} + \frac{1}{\delta} = 1; \\ \sum_{1 \leq i \neq j \leq n} |\alpha_i| |\alpha_j| \max_{1 \leq i \neq j \leq n} |(z_i, z_j)|, \end{cases} \tag{4.6}
\end{aligned}$$

$$= \begin{cases} \max_{1 \leq i \neq j \leq n} \{|\alpha_i \alpha_j|\} \sum_{1 \leq i \neq j \leq n} |(z_i, z_j)|; \\ \left[ \left( \sum_{i=1}^n |\alpha_i|^\gamma \right)^2 - \left( \sum_{i=1}^n |\alpha_i|^{2\gamma} \right) \right]^{\frac{1}{\gamma}} \left( \sum_{1 \leq i \neq j \leq n} |(z_i, z_j)|^\delta \right)^{\frac{1}{\delta}}, \\ \quad \text{where } \gamma > 1, \quad \frac{1}{\gamma} + \frac{1}{\delta} = 1; \\ \left[ \left( \sum_{i=1}^n |\alpha_i| \right)^2 - \sum_{i=1}^n |\alpha_i|^2 \right] \max_{1 \leq i \neq j \leq n} |(z_i, z_j)|. \end{cases}$$

Utilising (4.5) and (4.6) in (4.4), we may deduce the desired result (4.3). ■

**Remark 165** *Inequality (4.3) contains in fact 9 different inequalities which may be obtained by combining the first 3 ones with the last 3 ones.*

A particular case that may be related to the Boas-Bellman result is embodied in the following inequality [3].

**Corollary 166** *With the assumptions in Lemma 164, we have*

$$\begin{aligned} & \left\| \sum_{i=1}^n \alpha_i z_i \right\|^2 \\ & \leq \sum_{i=1}^n |\alpha_i|^2 \left\{ \max_{1 \leq i \leq n} \|z_i\|^2 + \frac{\left[ \left( \sum_{i=1}^n |\alpha_i|^2 \right)^2 - \sum_{i=1}^n |\alpha_i|^4 \right]^{\frac{1}{2}}}{\sum_{i=1}^n |\alpha_i|^2} \left( \sum_{1 \leq i \neq j \leq n} |(z_i, z_j)|^2 \right)^{\frac{1}{2}} \right\} \\ & \leq \sum_{i=1}^n |\alpha_i|^2 \left\{ \max_{1 \leq i \leq n} \|z_i\|^2 + \left( \sum_{1 \leq i \neq j \leq n} |(z_i, z_j)|^2 \right)^{\frac{1}{2}} \right\}. \end{aligned} \tag{4.7}$$

The first inequality follows by taking the third branch in the first curly bracket with the second branch in the second curly bracket for  $\gamma = \delta = 2$ .

The second inequality in (4.7) follows by the fact that

$$\left[ \left( \sum_{i=1}^n |\alpha_i|^2 \right)^2 - \sum_{i=1}^n |\alpha_i|^4 \right]^{\frac{1}{2}} \leq \sum_{i=1}^n |\alpha_i|^2.$$

Applying the following Cauchy-Bunyakovsky-Schwarz type inequality

$$\left( \sum_{i=1}^n a_i \right)^2 \leq n \sum_{i=1}^n a_i^2, \quad a_i \in \mathbb{R}_+, \quad 1 \leq i \leq n,$$



we may write that

$$\left( \sum_{i=1}^n |\alpha_i|^\gamma \right)^2 - \sum_{i=1}^n |\alpha_i|^{2\gamma} \leq (n-1) \sum_{i=1}^n |\alpha_i|^{2\gamma} \quad (n \geq 1) \quad (4.8)$$

and

$$\left( \sum_{i=1}^n |\alpha_i| \right)^2 - \sum_{i=1}^n |\alpha_i|^2 \leq (n-1) \sum_{i=1}^n |\alpha_i|^2 \quad (n \geq 1). \quad (4.9)$$

Also, it is obvious that:

$$\max_{1 \leq i \neq j \leq n} \{|\alpha_i \alpha_j|\} \leq \max_{1 \leq i \leq n} |\alpha_i|^2. \quad (4.10)$$

Consequently, we may state the following coarser upper bounds for  $\|\sum_{i=1}^n \alpha_i z_i\|^2$  that may be useful in applications [3].

**Corollary 167** *With the assumptions in Lemma 164, we have the inequalities:*

$$\begin{aligned} & \left\| \sum_{i=1}^n \alpha_i z_i \right\|^2 \\ & \leq \begin{cases} \max_{1 \leq i \leq n} |\alpha_i|^2 \sum_{i=1}^n \|z_i\|^2; \\ \left( \sum_{i=1}^n |\alpha_i|^{2\alpha} \right)^{\frac{1}{\alpha}} \left( \sum_{i=1}^n \|z_i\|^{2\beta} \right)^{\frac{1}{\beta}}, \quad \text{where } \alpha > 1, \frac{1}{\alpha} + \frac{1}{\beta} = 1; \\ \sum_{i=1}^n |\alpha_i|^2 \max_{1 \leq i \leq n} \|z_i\|^2, \end{cases} \\ & \quad + \begin{cases} \max_{1 \leq i \leq n} |\alpha_i|^2 \sum_{1 \leq i \neq j \leq n} |(z_i, z_j)|; \\ (n-1)^{\frac{1}{\gamma}} \left( \sum_{i=1}^n |\alpha_i|^{2\gamma} \right)^{\frac{1}{\gamma}} \left( \sum_{1 \leq i \neq j \leq n} |(z_i, z_j)|^\delta \right)^{\frac{1}{\delta}}, \\ \quad \text{where } \gamma > 1, \frac{1}{\gamma} + \frac{1}{\delta} = 1; \\ (n-1) \sum_{i=1}^n |\alpha_i|^2 \max_{1 \leq i \neq j \leq n} |(z_i, z_j)|. \end{cases} \quad (4.11) \end{aligned}$$

The proof is obvious by Lemma 164 in applying the inequalities (4.8) – (4.10).

**Remark 168** *The following inequalities which are incorporated in (4.11) are of special interest:*

$$\left\| \sum_{i=1}^n \alpha_i z_i \right\|^2 \leq \max_{1 \leq i \leq n} |\alpha_i|^2 \left[ \sum_{i=1}^n \|z_i\|^2 + \sum_{1 \leq i \neq j \leq n} |(z_i, z_j)| \right]; \quad (4.12)$$

$$\left\| \sum_{i=1}^n \alpha_i z_i \right\|^2 \leq \left( \sum_{i=1}^n |\alpha_i|^{2p} \right)^{\frac{1}{p}} \left[ \left( \sum_{i=1}^n \|z_i\|^{2q} \right)^{\frac{1}{q}} + (n-1)^{\frac{1}{p}} \left( \sum_{1 \leq i \neq j \leq n} |(z_i, z_j)|^q \right)^{\frac{1}{q}} \right], \quad (4.13)$$

where  $p > 1$ ,  $\frac{1}{p} + \frac{1}{q} = 1$ ; and

$$\left\| \sum_{i=1}^n \alpha_i z_i \right\|^2 \leq \sum_{i=1}^n |\alpha_i|^2 \left[ \max_{1 \leq i \leq n} \|z_i\|^2 + (n-1) \max_{1 \leq i \neq j \leq n} |(z_i, z_j)| \right]. \quad (4.14)$$

#### 4.1.3 Mitrinović-Pečarić-Fink Type Inequalities

We are now able to present the following result obtained in [3], which complements the inequality (4.2) due to Mitrinović, Pečarić and Fink [7, p. 392].

**Theorem 169** *Let  $x, y_1, \dots, y_n$  be vectors of an inner product space  $(H; (\cdot, \cdot))$  and  $c_1, \dots, c_n \in \mathbb{K}$  ( $\mathbb{K} = \mathbb{C}, \mathbb{R}$ ). Then one has the inequalities:*

$$\left| \sum_{i=1}^n c_i (x, y_i) \right|^2 \leq \|x\|^2 \begin{cases} \max_{1 \leq i \leq n} |c_i|^2 \sum_{i=1}^n \|y_i\|^2; \\ \left( \sum_{i=1}^n |c_i|^{2\alpha} \right)^{\frac{1}{\alpha}} \left( \sum_{i=1}^n \|y_i\|^{2\beta} \right)^{\frac{1}{\beta}}, & \text{where } \alpha > 1, \\ \sum_{i=1}^n |c_i|^2 \max_{1 \leq i \leq n} \|y_i\|^2, & \frac{1}{\alpha} + \frac{1}{\beta} = 1; \end{cases} \quad (4.15)$$

$$+ \|x\|^2 \begin{cases} \max_{1 \leq i \neq j \leq n} \{|c_i c_j|\} \sum_{1 \leq i \neq j \leq n} |(y_i, y_j)|; \\ \left[ \left( \sum_{i=1}^n |c_i|^\gamma \right)^2 - \left( \sum_{i=1}^n |c_i|^{2\gamma} \right) \right]^{\frac{1}{\gamma}} \\ \times \left( \sum_{1 \leq i \neq j \leq n} |(y_i, y_j)|^\delta \right)^{\frac{1}{\delta}}, & \text{where } \gamma > 1, \quad \frac{1}{\gamma} + \frac{1}{\delta} = 1; \\ \left[ \left( \sum_{i=1}^n |c_i| \right)^2 - \sum_{i=1}^n |c_i|^2 \right] \max_{1 \leq i \neq j \leq n} |(y_i, y_j)|. \end{cases}$$

**Proof.** We note that

$$\sum_{i=1}^n c_i (x, y_i) = \left( x, \sum_{i=1}^n \bar{c}_i y_i \right).$$

Using Schwarz's inequality in inner product spaces, we have:

$$\left| \sum_{i=1}^n c_i(x, y_i) \right|^2 \leq \|x\|^2 \left\| \sum_{i=1}^n \overline{c_i} y_i \right\|^2.$$

Now using Lemma 164 with  $\alpha_i = \overline{c_i}$ ,  $z_i = y_i$  ( $i = 1, \dots, n$ ), we deduce the desired inequality (4.15). ■

The following particular inequalities that may be obtained by the Corollaries 166 and 167 and Remark 168 hold [3].

**Corollary 170** *With the assumptions in Theorem 169, one has the inequalities:*

$$\left| \sum_{i=1}^n c_i(x, y_i) \right|^2 \leq \|x\|^2 \left\{ \begin{array}{l} \sum_{i=1}^n |c_i|^2 \left\{ \max_{1 \leq i \leq n} \|y_i\|^2 + \left( \sum_{1 \leq i \neq j \leq n} |(y_i, y_j)|^2 \right)^{\frac{1}{2}} \right\}; \\ \max_{1 \leq i \leq n} |c_i|^2 \left\{ \sum_{i=1}^n \|y_i\|^2 + \sum_{1 \leq i \neq j \leq n} |(y_i, y_j)| \right\} \\ \left( \sum_{i=1}^n |c_i|^{2p} \right)^{\frac{1}{p}} \left\{ \left( \sum_{i=1}^n \|y_i\|^{2q} \right)^{\frac{1}{q}} \right. \\ \quad \left. + (n-1)^{\frac{1}{p}} \left( \sum_{1 \leq i \neq j \leq n} |(y_i, y_j)|^q \right)^{\frac{1}{q}} \right\}, \\ \quad \text{where } p > 1, \frac{1}{p} + \frac{1}{q} = 1; \\ \sum_{i=1}^n |c_i|^2 \left\{ \max_{1 \leq i \leq n} \|y_i\|^2 + (n-1) \max_{1 \leq i \neq j \leq n} |(y_i, y_j)| \right\}. \end{array} \right. \quad (4.16)$$

**Remark 171** *Note that the first inequality in (4.16) is the result obtained by Mitrinović-Pečarić-Fink in [7]. The other 3 provide similar bounds in terms of the  $p$ -norms of the vector  $(|c_1|^2, \dots, |c_n|^2)$ .*

#### 4.1.4 Boas-Bellman Type Inequalities

If one chooses  $c_i = \overline{(x, y_i)}$  ( $i = 1, \dots, n$ ) in (4.15), then it is possible to obtain 9 different inequalities between the Fourier coefficients  $(x, y_i)$  and the norms and inner products of the vectors  $y_i$  ( $i = 1, \dots, n$ ). We restrict ourselves only to those inequalities that may be obtained from (4.16).

As Mitrinović, Pečarić and Fink noted in [7, p. 392], the first inequality in (4.16) for the above selection of  $c_i$  will produce the Boas-Bellman inequality (4.1).

From the second inequality in (4.16) for  $c_i = \overline{(x, y_i)}$  we get

$$\left( \sum_{i=1}^n |(x, y_i)|^2 \right)^2 \leq \|x\|^2 \max_{1 \leq i \leq n} |(x, y_i)|^2 \left\{ \sum_{i=1}^n \|y_i\|^2 + \sum_{1 \leq i \neq j \leq n} |(y_i, y_j)| \right\}.$$

Taking the square root in this inequality we obtain:

$$\begin{aligned} \sum_{i=1}^n |(x, y_i)|^2 &\leq \|x\| \max_{1 \leq i \leq n} |(x, y_i)| \left\{ \sum_{i=1}^n \|y_i\|^2 + \sum_{1 \leq i \neq j \leq n} |(y_i, y_j)| \right\}^{\frac{1}{2}}, \quad (4.17) \end{aligned}$$

for any  $x, y_1, \dots, y_n$  vectors in the inner product space  $(H; (\cdot, \cdot))$ .

If we assume that  $(e_i)_{1 \leq i \leq n}$  is an orthonormal family in  $H$ , then by (4.17) we have

$$\sum_{i=1}^n |(x, e_i)|^2 \leq \sqrt{n} \|x\| \max_{1 \leq i \leq n} |(x, e_i)|, \quad x \in H.$$

From the third inequality in (4.16) for  $c_i = \overline{(x, y_i)}$  we deduce

$$\begin{aligned} \left( \sum_{i=1}^n |(x, y_i)|^2 \right)^2 &\leq \|x\|^2 \left( \sum_{i=1}^n |(x, y_i)|^{2p} \right)^{\frac{1}{p}} \\ &\quad \times \left\{ \left( \sum_{i=1}^n \|y_i\|^{2q} \right)^{\frac{1}{q}} + (n-1)^{\frac{1}{p}} \left( \sum_{1 \leq i \neq j \leq n} |(y_i, y_j)|^q \right)^{\frac{1}{q}} \right\}, \end{aligned}$$

for  $p > 1, \frac{1}{p} + \frac{1}{q} = 1$ .

Taking the square root in this inequality we get

$$\begin{aligned} \sum_{i=1}^n |(x, y_i)|^2 &\leq \|x\| \left( \sum_{i=1}^n |(x, y_i)|^{2p} \right)^{\frac{1}{2p}} \\ &\quad \times \left\{ \left( \sum_{i=1}^n \|y_i\|^{2q} \right)^{\frac{1}{q}} + (n-1)^{\frac{1}{p}} \left( \sum_{1 \leq i \neq j \leq n} |(y_i, y_j)|^q \right)^{\frac{1}{q}} \right\}^{\frac{1}{2}}, \quad (4.18) \end{aligned}$$

for any  $x, y_1, \dots, y_n \in H, p > 1, \frac{1}{p} + \frac{1}{q} = 1$ .

The above inequality (4.18) becomes, for an orthonormal family  $(e_i)_{1 \leq i \leq n}$ ,

$$\sum_{i=1}^n |(x, e_i)|^2 \leq n^{\frac{1}{q}} \|x\| \left( \sum_{i=1}^n |(x, e_i)|^{2p} \right)^{\frac{1}{2p}}, \quad x \in H.$$

Finally, the choice  $c_i = \overline{(x, y_i)}$  ( $i = 1, \dots, n$ ) will produce in the last inequality in (4.16)

$$\left( \sum_{i=1}^n |(x, y_i)|^2 \right)^2 \leq \|x\|^2 \sum_{i=1}^n |(x, y_i)|^2 \left\{ \max_{1 \leq i \leq n} \|y_i\|^2 + (n-1) \max_{1 \leq i \neq j \leq n} |(y_i, y_j)| \right\}$$

giving the following Boas-Bellman type inequality

$$\sum_{i=1}^n |(x, y_i)|^2 \leq \|x\|^2 \left\{ \max_{1 \leq i \leq n} \|y_i\|^2 + (n-1) \max_{1 \leq i \neq j \leq n} |(y_i, y_j)| \right\}, \quad (4.19)$$

for any  $x, y_1, \dots, y_n \in H$ .

It is obvious that (4.19) will give for orthonormal families the well known Bessel inequality.

**Remark 172** *In order to compare the Boas-Bellman result with our result (4.19), it is enough to compare the quantities*

$$A := \left( \sum_{1 \leq i \neq j \leq n} |(y_i, y_j)|^2 \right)^{\frac{1}{2}}$$

and

$$B := (n-1) \max_{1 \leq i \neq j \leq n} |(y_i, y_j)|.$$

Consider the inner product space  $H = \mathbb{R}$  with  $(x, y) = xy$ , and choose  $n = 3$ ,  $y_1 = a > 0$ ,  $y_2 = b > 0$ ,  $y_3 = c > 0$ . Then

$$A = \sqrt{2} (a^2 b^2 + b^2 c^2 + c^2 a^2)^{\frac{1}{2}}, \quad B = 2 \max(ab, ac, bc).$$

Denote  $ab = p$ ,  $bc = q$ ,  $ca = r$ . Then

$$A = \sqrt{2} (p^2 + q^2 + r^2)^{\frac{1}{2}}, \quad B = 2 \max(p, q, r).$$

Firstly, if we assume that  $p = q = r$ , then  $A = \sqrt{6}p$ ,  $B = 2p$  which shows that  $A > B$ .

Now choose  $r = 1$  and  $p, q = \frac{1}{2}$ . Then  $A = \sqrt{3}$  and  $B = 2$  showing that  $B > A$ .

Consequently, in general, the Boas-Bellman inequality and our inequality (4.19) cannot be compared.

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## 4.2 Bombieri Type Inequalities

### 4.2.1 Introduction

In 1971, E. Bombieri [3] (see also [9, p. 394]) gave the following generalisation of Bessel's inequality.

**Theorem 173** *If  $x, y_1, \dots, y_n$  are vectors in the inner product space  $(H; (\cdot, \cdot))$ , then the following inequality:*

$$\sum_{i=1}^n |(x, y_i)|^2 \leq \|x\|^2 \max_{1 \leq i \leq n} \left\{ \sum_{j=1}^n |(y_i, y_j)| \right\}, \quad (4.20)$$

*holds.*

It is obvious that if  $(y_i)_{1 \leq i \leq n}$  are orthonormal, then from (4.20) one can deduce Bessel's inequality.

Another generalisation of Bessel's inequality was obtained by A. Selberg (see for example [9, p. 394]):

**Theorem 174** *Let  $x, y_1, \dots, y_n$  be vectors in  $H$  with  $y_i \neq 0$  ( $i = 1, \dots, n$ ). Then one has the inequality:*

$$\sum_{i=1}^n \frac{|(x, y_i)|^2}{\sum_{j=1}^n |(y_i, y_j)|} \leq \|x\|^2. \quad (4.21)$$

In this case, also, if  $(y_i)_{1 \leq i \leq n}$  are orthonormal, then from (4.21) one may deduce Bessel's inequality.

Another type of inequality related to Bessel's result, was discovered in 1958 by H. Heilbronn [8] (see also [9, p. 395]).

**Theorem 175** *With the assumptions in Theorem 173, one has*

$$\sum_{i=1}^n |(x, y_i)| \leq \|x\| \left( \sum_{i,j=1}^n |(y_i, y_j)| \right)^{\frac{1}{2}}. \quad (4.22)$$

If in (4.22) one chooses  $y_i = e_i$  ( $i = 1, \dots, n$ ), where  $(e_i)_{1 \leq i \leq n}$  are orthonormal vectors in  $H$ , then

$$\sum_{i=1}^n |(x, e_i)| \leq \sqrt{n} \|x\|, \quad \text{for any } x \in H.$$

In 1992, J.E. Pečarić [10] (see also [9, p. 394]) proved the following general inequality in inner product spaces.

**Theorem 176** *Let  $x, y_1, \dots, y_n \in H$  and  $c_1, \dots, c_n \in \mathbb{K}$ . Then*

$$\begin{aligned} \left| \sum_{i=1}^n c_i (x, y_i) \right|^2 &\leq \|x\|^2 \sum_{i=1}^n |c_i|^2 \left( \sum_{j=1}^n |(y_i, y_j)| \right) \\ &\leq \|x\|^2 \sum_{i=1}^n |c_i|^2 \max_{1 \leq i \leq n} \left\{ \sum_{j=1}^n |(y_i, y_j)| \right\}. \end{aligned} \quad (4.23)$$

He showed that the Bombieri inequality (4.20) may be obtained from (4.23) for the choice  $c_i = \overline{(x, y_i)}$  (using the second inequality), the Selberg inequality (4.21) may be obtained from the first part of (4.23) for the choice

$$c_i = \frac{\overline{(x, y_i)}}{\sum_{j=1}^n |(y_i, y_j)|}, \quad i \in \{1, \dots, n\};$$

while the Heilbronn inequality (4.22) may be obtained from the first part of (4.23) if one chooses  $c_i = \frac{\overline{(x, y_i)}}{|(x, y_i)|}$ , for any  $i \in \{1, \dots, n\}$ .

For other results connected with the above ones, see [6] and [7].

#### 4.2.2 Some Norm Inequalities

We start with the following lemma which is also interesting in itself [4].

**Lemma 177** *Let  $z_1, \dots, z_n \in H$  and  $\alpha_1, \dots, \alpha_n \in \mathbb{K}$ . Then one has the inequality:*

$$\left\| \sum_{i=1}^n \alpha_i z_i \right\|^2 \leq \begin{Bmatrix} A \\ B \\ C \end{Bmatrix}, \quad (4.24)$$

where

$$A := \begin{cases} \max_{1 \leq k \leq n} |\alpha_k|^2 \sum_{i,j=1}^n |(z_i, z_j)|; \\ \max_{1 \leq k \leq n} |\alpha_k| \left( \sum_{i=1}^n |\alpha_i|^r \right)^{\frac{1}{r}} \left( \sum_{i=1}^n \left( \sum_{j=1}^n |(z_i, z_j)| \right)^s \right)^{\frac{1}{s}}, & r > 1, \frac{1}{r} + \frac{1}{s} = 1; \\ \max_{1 \leq k \leq n} |\alpha_k| \sum_{k=1}^n |\alpha_k| \max_{1 \leq i \leq n} \left( \sum_{j=1}^n |(z_i, z_j)| \right); \end{cases}$$



$$B := \begin{cases} \left( \sum_{k=1}^n |\alpha_k|^p \right)^{\frac{1}{p}} \max_{1 \leq i \leq n} |\alpha_i| \left( \sum_{i=1}^n \left( \sum_{j=1}^n |(z_i, z_j)| \right)^q \right)^{\frac{1}{q}}, & p > 1, \frac{1}{p} + \frac{1}{q} = 1; \\ \left( \sum_{k=1}^n |\alpha_k|^p \right)^{\frac{1}{p}} \left( \sum_{i=1}^n |\alpha_i|^t \right)^{\frac{1}{t}} \left[ \sum_{i=1}^n \left( \sum_{j=1}^n |(z_i, z_j)|^q \right)^{\frac{u}{q}} \right]^{\frac{1}{u}}, & p > 1, \frac{1}{p} + \frac{1}{q} = 1; \\ & t > 1, \frac{1}{t} + \frac{1}{u} = 1; \\ \left( \sum_{k=1}^n |\alpha_k|^p \right)^{\frac{1}{p}} \sum_{i=1}^n |\alpha_i| \max_{1 \leq i \leq n} \left\{ \left( \sum_{j=1}^n |(z_i, z_j)|^q \right)^{\frac{1}{q}} \right\}, & p > 1, \frac{1}{p} + \frac{1}{q} = 1; \end{cases}$$

and

$$C := \begin{cases} \sum_{k=1}^n |\alpha_k| \max_{1 \leq i \leq n} |\alpha_i| \sum_{i=1}^n \left[ \max_{1 \leq j \leq n} |(z_i, z_j)| \right]; \\ \sum_{k=1}^n |\alpha_k| \left( \sum_{i=1}^n |\alpha_i|^m \right)^{\frac{1}{m}} \left( \sum_{i=1}^n \left[ \max_{1 \leq j \leq n} |(z_i, z_j)| \right]^l \right)^{\frac{1}{l}}, & m > 1, \frac{1}{m} + \frac{1}{l} = 1; \\ \left( \sum_{k=1}^n |\alpha_k| \right)^2 \max_{i, 1 \leq j \leq n} |(z_i, z_j)|. \end{cases}$$

**Proof.** We observe that

$$\begin{aligned} \left\| \sum_{i=1}^n \alpha_i z_i \right\|^2 &= \left( \sum_{i=1}^n \alpha_i z_i, \sum_{j=1}^n \alpha_j z_j \right) \\ &= \sum_{i=1}^n \sum_{j=1}^n \alpha_i \overline{\alpha_j} (z_i, z_j) = \left| \sum_{i=1}^n \sum_{j=1}^n \alpha_i \overline{\alpha_j} (z_i, z_j) \right| \\ &\leq \sum_{i=1}^n \sum_{j=1}^n |\alpha_i| |\alpha_j| |(z_i, z_j)| = \sum_{i=1}^n |\alpha_i| \left( \sum_{j=1}^n |\alpha_j| |(z_i, z_j)| \right) \\ &:= M. \end{aligned}$$

Using Hölder's inequality, we may write that

$$\sum_{j=1}^n |\alpha_j| |(z_i, z_j)| \leq \begin{cases} \max_{1 \leq k \leq n} |\alpha_k| \sum_{j=1}^n |(z_i, z_j)| \\ \left( \sum_{k=1}^n |\alpha_k|^p \right)^{\frac{1}{p}} \left( \sum_{j=1}^n |(z_i, z_j)|^q \right)^{\frac{1}{q}}, & p > 1, \frac{1}{p} + \frac{1}{q} = 1; \\ \sum_{k=1}^n |\alpha_k| \max_{1 \leq j \leq n} |(z_i, z_j)| \end{cases}$$

for any  $i \in \{1, \dots, n\}$ , giving

$$M \leq \begin{cases} \max_{1 \leq k \leq n} |\alpha_k| \sum_{i=1}^n |\alpha_i| \sum_{j=1}^n |(z_i, z_j)| =: M_1; \\ \left( \sum_{k=1}^n |\alpha_k|^p \right)^{\frac{1}{p}} \sum_{i=1}^n |\alpha_i| \left( \sum_{j=1}^n |(z_i, z_j)|^q \right)^{\frac{1}{q}} := M_p, \\ \quad p > 1, \frac{1}{p} + \frac{1}{q} = 1; \\ \sum_{k=1}^n |\alpha_k| \sum_{i=1}^n |\alpha_i| \max_{1 \leq j \leq n} |(z_i, z_j)| =: M_\infty. \end{cases}$$

By Hölder's inequality we also have:

$$\begin{aligned} & \sum_{i=1}^n |\alpha_i| \left( \sum_{j=1}^n |(z_i, z_j)| \right) \\ & \leq \begin{cases} \max_{1 \leq i \leq n} |\alpha_i| \sum_{i,j=1}^n |(z_i, z_j)|; \\ \left( \sum_{i=1}^n |\alpha_i|^r \right)^{\frac{1}{r}} \left( \sum_{i=1}^n \left( \sum_{j=1}^n |(z_i, z_j)| \right)^s \right)^{\frac{1}{s}}, \quad r > 1, \frac{1}{r} + \frac{1}{s} = 1; \\ \sum_{i=1}^n |\alpha_i| \max_{1 \leq i \leq n} \left( \sum_{j=1}^n |(z_i, z_j)| \right); \end{cases} \end{aligned}$$

and thus

$$M_1 \leq \begin{cases} \max_{1 \leq k \leq n} |\alpha_k|^2 \sum_{i,j=1}^n |(z_i, z_j)|; \\ \max_{1 \leq k \leq n} |\alpha_k| \left( \sum_{i=1}^n |\alpha_i|^r \right)^{\frac{1}{r}} \left( \sum_{i=1}^n \left( \sum_{j=1}^n |(z_i, z_j)| \right)^s \right)^{\frac{1}{s}}, \quad r > 1, \frac{1}{r} + \frac{1}{s} = 1; \\ \max_{1 \leq k \leq n} |\alpha_k| \sum_{i=1}^n |\alpha_i| \max_{1 \leq i \leq n} \left( \sum_{j=1}^n |(z_i, z_j)| \right); \end{cases}$$

and the first 3 inequalities in (4.24) are obtained.

By Hölder's inequality we also have:

$$M_p \leq \left( \sum_{k=1}^n |\alpha_k|^p \right)^{\frac{1}{p}} \times \begin{cases} \max_{1 \leq i \leq n} |\alpha_i| \sum_{i=1}^n \left( \sum_{j=1}^n |(z_i, z_j)|^q \right)^{\frac{1}{q}} ; \\ \left( \sum_{i=1}^n |\alpha_i|^t \right)^{\frac{1}{t}} \left( \sum_{i=1}^n \left( \sum_{j=1}^n |(z_i, z_j)|^q \right)^{\frac{u}{q}} \right)^{\frac{1}{u}} , & t > 1, \frac{1}{t} + \frac{1}{u} = 1; \\ \sum_{i=1}^n |\alpha_i| \max_{1 \leq i \leq n} \left\{ \left( \sum_{j=1}^n |(z_i, z_j)|^q \right)^{\frac{1}{q}} \right\} ; \end{cases}$$

and the next 3 inequalities in (4.24) are proved.

Finally, by the same Hölder inequality we may state that:

$$M_\infty \leq \sum_{k=1}^n |\alpha_k| \times \begin{cases} \max_{1 \leq i \leq n} |\alpha_i| \sum_{i=1}^n \left( \max_{1 \leq j \leq n} |(z_i, z_j)| \right) ; \\ \left( \sum_{i=1}^n |\alpha_i|^m \right)^{\frac{1}{m}} \left( \sum_{i=1}^n \left( \max_{1 \leq j \leq n} |(z_i, z_j)| \right)^l \right)^{\frac{1}{l}} , & m > 1, \frac{1}{m} + \frac{1}{l} = 1; \\ \sum_{i=1}^n |\alpha_i| \max_{1 \leq i, j \leq n} |(z_i, z_j)| ; \end{cases}$$

and the last 3 inequalities in (4.24) are proved. ■

If we would like to have some bounds for  $\|\sum_{i=1}^n \alpha_i z_i\|^2$  in terms of  $\sum_{i=1}^n |\alpha_i|^2$ , then the following corollaries may be used.

**Corollary 178** *Let  $z_1, \dots, z_n$  and  $\alpha_1, \dots, \alpha_n$  be as in Lemma 177. If  $1 < p \leq 2$ ,  $1 < t \leq 2$ , then one has the inequality*

$$\left\| \sum_{i=1}^n \alpha_i z_i \right\|^2 \leq n^{\frac{1}{p} + \frac{1}{t} - 1} \sum_{k=1}^n |\alpha_k|^2 \left[ \sum_{i=1}^n \left( \sum_{j=1}^n |(z_i, z_j)|^q \right)^{\frac{u}{q}} \right]^{\frac{1}{u}} , \quad (4.25)$$

where  $\frac{1}{p} + \frac{1}{q} = 1$ ,  $\frac{1}{t} + \frac{1}{u} = 1$ .

**Proof.** Observe, by the monotonicity of power means, we may write that

$$\begin{aligned} \left( \frac{\sum_{k=1}^n |\alpha_k|^p}{n} \right)^{\frac{1}{p}} &\leq \left( \frac{\sum_{k=1}^n |\alpha_k|^2}{n} \right)^{\frac{1}{2}} ; \quad 1 < p \leq 2, \\ \left( \frac{\sum_{k=1}^n |\alpha_k|^t}{n} \right)^{\frac{1}{t}} &\leq \left( \frac{\sum_{k=1}^n |\alpha_k|^2}{n} \right)^{\frac{1}{2}} ; \quad 1 < t \leq 2, \end{aligned}$$

from where we get

$$\begin{aligned} \left( \sum_{k=1}^n |\alpha_k|^p \right)^{\frac{1}{p}} &\leq n^{\frac{1}{p}-\frac{1}{2}} \left( \sum_{k=1}^n |\alpha_k|^2 \right)^{\frac{1}{2}}, \\ \left( \sum_{k=1}^n |\alpha_k|^t \right)^{\frac{1}{t}} &\leq n^{\frac{1}{t}-\frac{1}{2}} \left( \sum_{k=1}^n |\alpha_k|^2 \right)^{\frac{1}{2}}. \end{aligned}$$

Using the fifth inequality in (4.24), we then deduce (4.25). ■

**Remark 179** *An interesting particular case is the one for  $p = q = t = u = 2$ , giving*

$$\left\| \sum_{i=1}^n \alpha_i z_i \right\|^2 \leq \sum_{k=1}^n |\alpha_k|^2 \left( \sum_{i,j=1}^n |(z_i, z_j)|^2 \right)^{\frac{1}{2}}.$$

**Corollary 180** *With the assumptions of Lemma 177 and if  $1 < p \leq 2$ , then*

$$\left\| \sum_{i=1}^n \alpha_i z_i \right\|^2 \leq n^{\frac{1}{p}} \sum_{k=1}^n |\alpha_k|^2 \max_{1 \leq i \leq n} \left[ \left( \sum_{j=1}^n |(z_i, z_j)|^q \right)^{\frac{1}{q}} \right], \quad (4.26)$$

where  $\frac{1}{p} + \frac{1}{q} = 1$ .

**Proof.** Since

$$\left( \sum_{k=1}^n |\alpha_k|^p \right)^{\frac{1}{p}} \leq n^{\frac{1}{p}-\frac{1}{2}} \left( \sum_{k=1}^n |\alpha_k|^2 \right)^{\frac{1}{2}},$$

and

$$\sum_{k=1}^n |\alpha_k| \leq n^{\frac{1}{2}} \left( \sum_{k=1}^n |\alpha_k|^2 \right)^{\frac{1}{2}},$$

then by the sixth inequality in (4.24) we deduce (4.26). ■

In a similar fashion, one may prove the following two corollaries.

**Corollary 181** *With the assumptions of Lemma 177 and if  $1 < m \leq 2$ , then*

$$\left\| \sum_{i=1}^n \alpha_i z_i \right\|^2 \leq n^{\frac{1}{m}} \sum_{k=1}^n |\alpha_k|^2 \left( \sum_{i=1}^n \left[ \max_{1 \leq j \leq n} |(z_i, z_j)| \right]^l \right)^{\frac{1}{l}},$$

where  $\frac{1}{m} + \frac{1}{l} = 1$ .

**Corollary 182** *With the assumptions of Lemma 177, we have:*

$$\left\| \sum_{i=1}^n \alpha_i z_i \right\|^2 \leq n \sum_{k=1}^n |\alpha_k|^2 \max_{1 \leq i, j \leq n} |(z_i, z_j)|.$$

The following lemma may be of interest as well [4].

**Lemma 183** *With the assumptions of Lemma 177, one has the inequalities*

$$\begin{aligned} \left\| \sum_{i=1}^n \alpha_i z_i \right\|^2 &\leq \sum_{i=1}^n |\alpha_i|^2 \sum_{j=1}^n |(z_i, z_j)| \\ &\leq \begin{cases} \sum_{i=1}^n |\alpha_i|^2 \max_{1 \leq i \leq n} \left[ \sum_{j=1}^n |(z_i, z_j)| \right]; \\ \left( \sum_{i=1}^n |\alpha_i|^{2p} \right)^{\frac{1}{p}} \left( \left( \sum_{j=1}^n |(z_i, z_j)| \right)^q \right)^{\frac{1}{q}}, \\ \quad p > 1, \frac{1}{p} + \frac{1}{q} = 1; \\ \max_{1 \leq i \leq n} |\alpha_i|^2 \sum_{i,j=1}^n |(z_i, z_j)|. \end{cases} \end{aligned} \quad (4.27)$$

**Proof.** As in Lemma 177, we know that

$$\left\| \sum_{i=1}^n \alpha_i z_i \right\|^2 \leq \sum_{i=1}^n \sum_{j=1}^n |\alpha_i| |\alpha_j| |(z_i, z_j)|.$$

Using the simple observation that (see also [9, p. 394])

$$|\alpha_i| |\alpha_j| \leq \frac{1}{2} (|\alpha_i|^2 + |\alpha_j|^2), \quad i, j \in \{1, \dots, n\},$$

we have

$$\begin{aligned} \sum_{i=1}^n \sum_{j=1}^n |\alpha_i| |\alpha_j| |(z_i, z_j)| &\leq \frac{1}{2} \sum_{i,j=1}^n (|\alpha_i|^2 + |\alpha_j|^2) |(z_i, z_j)| \\ &= \frac{1}{2} \left[ \sum_{i,j=1}^n |\alpha_i|^2 |(z_i, z_j)| + \sum_{i,j=1}^n |\alpha_j|^2 |(z_i, z_j)| \right] \\ &= \sum_{i,j=1}^n |\alpha_i|^2 |(z_i, z_j)|, \end{aligned}$$

which proves the first inequality in (4.27).

The second part follows by Hölder's inequality and we omit the details. ■

**Remark 184** *The first part in (4.27) is the inequality obtained by Pečarić in [10].*

### 4.2.3 Pečarić Type Inequalities

We are now able to present the following result obtained in [4], which complements the inequality (4.23) due to J.E. Pečarić [10] (see also [9, p. 394]).

**Theorem 185** *Let  $x, y_1, \dots, y_n$  be vectors of an inner product space  $(H; (\cdot, \cdot))$  and  $c_1, \dots, c_n \in \mathbb{K}$ . Then one has the inequalities:*

$$\left| \sum_{i=1}^n c_i (x, y_i) \right|^2 \leq \|x\|^2 \times \begin{cases} D \\ E \\ F \end{cases}, \quad (4.28)$$

where

$$D := \begin{cases} \max_{1 \leq k \leq n} |c_k|^2 \sum_{i,j=1}^n |(y_i, y_j)|; \\ \max_{1 \leq k \leq n} |c_k| \left( \sum_{i=1}^n |c_i|^r \right)^{\frac{1}{r}} \left[ \sum_{i=1}^n \left( \sum_{j=1}^n |(y_i, y_j)| \right)^s \right]^{\frac{1}{s}}, & r > 1, \frac{1}{r} + \frac{1}{s} = 1; \\ \max_{1 \leq k \leq n} |c_k| \sum_{k=1}^n |c_k| \max_{1 \leq i \leq n} \left( \sum_{j=1}^n |(y_i, y_j)| \right); \end{cases}$$

$$E := \begin{cases} \left( \sum_{k=1}^n |c_k|^p \right)^{\frac{1}{p}} \max_{1 \leq i \leq n} |c_i| \left( \sum_{i=1}^n \left( \sum_{j=1}^n |(y_i, y_j)| \right)^q \right)^{\frac{1}{q}}, & p > 1, \frac{1}{p} + \frac{1}{q} = 1; \\ \left( \sum_{k=1}^n |c_k|^p \right)^{\frac{1}{p}} \left( \sum_{i=1}^n |c_i|^t \right)^{\frac{1}{t}} \left[ \sum_{i=1}^n \left( \sum_{j=1}^n |(y_i, y_j)|^q \right)^{\frac{u}{q}} \right]^{\frac{1}{u}}, & p > 1, \frac{1}{p} + \frac{1}{q} = 1; \\ & t > 1, \frac{1}{t} + \frac{1}{u} = 1; \\ \left( \sum_{k=1}^n |c_k|^p \right)^{\frac{1}{p}} \sum_{i=1}^n |c_i| \max_{1 \leq i \leq n} \left\{ \left( \sum_{j=1}^n |(y_i, y_j)|^q \right)^{\frac{1}{q}} \right\}, & p > 1, \frac{1}{p} + \frac{1}{q} = 1; \end{cases}$$

and

$$F := \begin{cases} \sum_{k=1}^n |c_k| \max_{1 \leq i \leq n} |c_i| \sum_{i=1}^n \left[ \max_{1 \leq j \leq n} |(y_i, y_j)| \right]; \\ \sum_{k=1}^n |c_k| \left( \sum_{i=1}^n |c_i|^m \right)^{\frac{1}{m}} \left( \sum_{i=1}^n \left[ \max_{1 \leq j \leq n} |(y_i, y_j)| \right]^l \right)^{\frac{1}{l}}, \\ \quad m > 1, \frac{1}{m} + \frac{1}{l} = 1; \\ \left( \sum_{k=1}^n |c_k| \right)^2 \max_{i, 1 \leq j \leq n} |(y_i, y_j)|. \end{cases}$$

**Proof.** We note that

$$\sum_{i=1}^n c_i (x, y_i) = \left( x, \sum_{i=1}^n \bar{c}_i y_i \right).$$

Using Schwarz's inequality in inner product spaces, we have

$$\left| \sum_{i=1}^n c_i (x, y_i) \right|^2 \leq \|x\|^2 \left\| \sum_{i=1}^n \bar{c}_i y_i \right\|^2.$$

Finally, using Lemma 177 with  $\alpha_i = \bar{c}_i$ ,  $z_i = y_i$  ( $i = 1, \dots, n$ ), we deduce the desired inequality (4.28). We omit the details. ■

The following corollaries may be useful if one needs bounds in terms of  $\sum_{i=1}^n |c_i|^2$ .

**Corollary 186** *With the assumptions in Theorem 185 and if  $1 < p \leq 2$ ,  $1 < t \leq 2$ ,  $\frac{1}{p} + \frac{1}{q} = 1$ ,  $\frac{1}{t} + \frac{1}{u} = 1$ , one has the inequality:*

$$\begin{aligned} & \left| \sum_{i=1}^n c_i (x, y_i) \right|^2 \\ & \leq \|x\|^2 n^{\frac{1}{p} + \frac{1}{t} - 1} \sum_{i=1}^n |c_i|^2 \left[ \sum_{i=1}^n \left( \sum_{j=1}^n |(y_i, y_j)|^q \right)^{\frac{u}{q}} \right]^{\frac{1}{u}}, \quad (4.29) \end{aligned}$$

and, in particular, for  $p = q = t = u = 2$ ,

$$\left| \sum_{i=1}^n c_i (x, y_i) \right|^2 \leq \|x\|^2 \sum_{i=1}^n |c_i|^2 \left( \sum_{i,j=1}^n |(y_i, y_j)|^2 \right)^{\frac{1}{2}}.$$

The proof is similar to the one in Corollary 178.

**Corollary 187** *With the assumptions in Theorem 185 and if  $1 < p \leq 2$ , then*

$$\left| \sum_{i=1}^n c_i(x, y_i) \right|^2 \leq \|x\|^2 n^{\frac{1}{p}} \sum_{k=1}^n |c_k|^2 \max_{1 \leq i \leq n} \left[ \sum_{j=1}^n |(y_i, y_j)|^q \right]^{\frac{1}{q}},$$

where  $\frac{1}{p} + \frac{1}{q} = 1$ .

The proof is similar to the one in Corollary 180.

The following two inequalities also hold.

**Corollary 188** *With the above assumptions for  $x, y_i, c_i$  and if  $1 < m \leq 2$ , then*

$$\left| \sum_{i=1}^n c_i(x, y_i) \right|^2 \leq \|x\|^2 n^{\frac{1}{m}} \sum_{k=1}^n |c_k|^2 \left( \sum_{i=1}^n \left[ \max_{1 \leq j \leq n} |(y_i, y_j)| \right]^l \right)^{\frac{1}{l}}, \quad (4.30)$$

where  $\frac{1}{m} + \frac{1}{l} = 1$ .

**Corollary 189** *With the above assumptions for  $x, y_i, c_i$ , one has*

$$\left| \sum_{i=1}^n c_i(x, y_i) \right|^2 \leq \|x\|^2 n \sum_{k=1}^n |c_k|^2 \max_{1 \leq j \leq n} |(y_i, y_j)|. \quad (4.31)$$

Using Lemma 183, we may state the following result as well.

**Remark 190** *With the assumptions of Theorem 185, one has the inequalities:*

$$\begin{aligned} \left| \sum_{i=1}^n c_i(x, y_i) \right|^2 &\leq \|x\|^2 \sum_{i=1}^n |c_i|^2 \sum_{j=1}^n |(y_i, y_j)| \\ &\leq \|x\|^2 \begin{cases} \sum_{i=1}^n |c_i|^2 \max_{1 \leq i \leq n} \left[ \sum_{j=1}^n |(y_i, y_j)| \right]; \\ \left( \sum_{i=1}^n |c_i|^{2p} \right)^{\frac{1}{p}} \left( \sum_{i=1}^n \left( \sum_{j=1}^n |(y_i, y_j)| \right)^q \right)^{\frac{1}{q}}, \\ \quad p > 1, \frac{1}{p} + \frac{1}{q} = 1; \\ \max_{1 \leq i \leq n} |c_i|^2 \sum_{i,j=1}^n |(y_i, y_j)|; \end{cases} \end{aligned}$$

that provide some alternatives to Pečarić's result (4.23).

#### 4.2.4 Inequalities of Bombieri Type

In this section we point out some inequalities of Bombieri type that may be obtained from (4.28) on choosing  $c_i = (x, y_i)$  ( $i = 1, \dots, n$ ).



If the above choice was made in the first inequality in (4.28), then one can obtain:

$$\left( \sum_{i=1}^n |(x, y_i)|^2 \right)^2 \leq \|x\|^2 \max_{1 \leq i \leq n} |(x, y_i)|^2 \sum_{i,j=1}^n |(y_i, y_j)|$$

giving, by taking the square root,

$$\sum_{i=1}^n |(x, y_i)|^2 \leq \|x\| \max_{1 \leq i \leq n} |(x, y_i)| \left( \sum_{i,j=1}^n |(y_i, y_j)| \right)^{\frac{1}{2}}, \quad x \in H. \quad (4.32)$$

If the same choice for  $c_i$  is made in the second inequality in (4.28), then one can get

$$\begin{aligned} \left( \sum_{i=1}^n |(x, y_i)|^2 \right)^2 &\leq \|x\|^2 \max_{1 \leq i \leq n} |(x, y_i)| \left( \sum_{i=1}^n |(x, y_i)|^r \right)^{\frac{1}{r}} \\ &\quad \times \left[ \sum_{i=1}^n \left( \sum_{j=1}^n |(y_i, y_j)| \right)^s \right]^{\frac{1}{s}}, \end{aligned}$$

implying

$$\begin{aligned} \sum_{i=1}^n |(x, y_i)|^2 &\leq \|x\| \max_{1 \leq i \leq n} |(x, y_i)|^{\frac{1}{2}} \left( \sum_{i=1}^n |(x, y_i)|^r \right)^{\frac{1}{2r}} \\ &\quad \times \left[ \sum_{i=1}^n \left( \sum_{j=1}^n |(y_i, y_j)| \right)^s \right]^{\frac{1}{2s}}, \quad (4.33) \end{aligned}$$

where  $\frac{1}{r} + \frac{1}{s} = 1$ ,  $s > 1$ .

The other inequalities in (4.28) will produce the following results, respectively

$$\begin{aligned} \sum_{i=1}^n |(x, y_i)|^2 &\leq \|x\| \max_{1 \leq i \leq n} |(x, y_i)|^{\frac{1}{2}} \left( \sum_{i=1}^n |(x, y_i)| \right)^{\frac{1}{2}} \\ &\quad \times \left[ \max_{1 \leq i \leq n} \left( \sum_{j=1}^n |(y_i, y_j)| \right) \right]; \quad (4.34) \end{aligned}$$

$$\sum_{i=1}^n |(x, y_i)|^2 \leq \|x\| \max_{1 \leq i \leq n} |(x, y_i)|^{\frac{1}{2}} \left( \sum_{i=1}^n |(x, y_i)|^p \right)^{\frac{1}{2p}} \times \left[ \sum_{i=1}^n \left( \sum_{j=1}^n |(y_i, y_j)|^q \right)^{\frac{1}{q}} \right]^{\frac{1}{2}}, \quad (4.35)$$

where  $p > 1$ ,  $\frac{1}{p} + \frac{1}{q} = 1$ ;

$$\sum_{i=1}^n |(x, y_i)|^2 \leq \|x\| \left( \sum_{i=1}^n |(x, y_i)|^p \right)^{\frac{1}{2p}} \left( \sum_{i=1}^n |(x, y_i)|^t \right)^{\frac{1}{2t}} \times \left[ \sum_{i=1}^n \left( \sum_{j=1}^n |(y_i, y_j)|^q \right)^{\frac{u}{q}} \right]^{\frac{1}{2u}}, \quad (4.36)$$

where  $p > 1$ ,  $\frac{1}{p} + \frac{1}{q} = 1$ ,  $t > 1$ ,  $\frac{1}{t} + \frac{1}{u} = 1$ ;

$$\sum_{i=1}^n |(x, y_i)|^2 \leq \|x\| \left( \sum_{i=1}^n |(x, y_i)|^p \right)^{\frac{1}{2p}} \left( \sum_{i=1}^n |(x, y_i)| \right)^{\frac{1}{2}} \times \max_{1 \leq i \leq n} \left\{ \left( \sum_{j=1}^n |(y_i, y_j)|^q \right)^{\frac{1}{2q}} \right\}, \quad (4.37)$$

where  $p > 1$ ,  $\frac{1}{p} + \frac{1}{q} = 1$ ;

$$\sum_{i=1}^n |(x, y_i)|^2 \leq \|x\| \left[ \sum_{i=1}^n |(x, y_i)| \right]^{\frac{1}{2}} \max_{1 \leq i \leq n} |(x, y_i)|^{\frac{1}{2}} \times \left( \sum_{i=1}^n \left[ \max_{1 \leq j \leq n} |(y_i, y_j)| \right] \right)^{\frac{1}{2}}; \quad (4.38)$$

$$\sum_{i=1}^n |(x, y_i)|^2 \leq \|x\| \left[ \sum_{i=1}^n |(x, y_i)|^m \right]^{\frac{1}{2m}} \left[ \sum_{i=1}^n \left[ \max_{1 \leq j \leq n} |(y_i, y_j)|^l \right] \right]^{\frac{1}{2l}}, \quad (4.39)$$

where  $m > 1$ ,  $\frac{1}{m} + \frac{1}{l} = 1$ ; and

$$\sum_{i=1}^n |(x, y_i)|^2 \leq \|x\| \sum_{i=1}^n |(x, y_i)| \max_{1 \leq j \leq n} |(y_i, y_j)|^{\frac{1}{2}}. \quad (4.40)$$

If in the above inequalities we assume that  $(y_i)_{1 \leq i \leq n} = (e_i)_{1 \leq i \leq n}$ , where  $(e_i)_{1 \leq i \leq n}$  are orthonormal vectors in the inner product space  $(H, (\cdot, \cdot))$ , then

from (4.32) – (4.40) we may deduce the following inequalities similar in a sense to Bessel's inequality:

$$\sum_{i=1}^n |(x, e_i)|^2 \leq \sqrt{n} \|x\| \max_{1 \leq i \leq n} \{|(x, e_i)|\};$$

$$\sum_{i=1}^n |(x, e_i)|^2 \leq n^{\frac{1}{2s}} \|x\| \max_{1 \leq i \leq n} \{|(x, e_i)|^{\frac{1}{2}}\} \left( \sum_{i=1}^n |(x, e_i)|^r \right)^{\frac{1}{2r}},$$

where  $r > 1$ ,  $\frac{1}{r} + \frac{1}{s} = 1$ ;

$$\sum_{i=1}^n |(x, e_i)|^2 \leq \|x\| \max_{1 \leq i \leq n} \{|(x, e_i)|^{\frac{1}{2}}\} \left( \sum_{i=1}^n |(x, e_i)| \right)^{\frac{1}{2}},$$

$$\sum_{i=1}^n |(x, e_i)|^2 \leq \sqrt{n} \|x\| \max_{1 \leq i \leq n} \{|(x, e_i)|^{\frac{1}{2}}\} \left( \sum_{i=1}^n |(x, e_i)|^p \right)^{\frac{1}{2p}},$$

where  $p > 1$ ;

$$\sum_{i=1}^n |(x, e_i)|^2 \leq n^{\frac{1}{2u}} \|x\| \left( \sum_{i=1}^n |(x, e_i)|^p \right)^{\frac{1}{2p}} \left( \sum_{i=1}^n |(x, e_i)|^t \right)^{\frac{1}{2t}},$$

where  $p > 1$ ,  $t > 1$ ,  $\frac{1}{t} + \frac{1}{u} = 1$ ;

$$\sum_{i=1}^n |(x, e_i)|^2 \leq \|x\| \left( \sum_{i=1}^n |(x, e_i)|^p \right)^{\frac{1}{2p}} \left( \sum_{i=1}^n |(x, e_i)| \right)^{\frac{1}{2}}, \quad p > 1;$$

$$\sum_{i=1}^n |(x, e_i)|^2 \leq \sqrt{n} \|x\| \left( \sum_{i=1}^n |(x, e_i)| \right)^{\frac{1}{2}} \max_{1 \leq i \leq n} \{|(x, e_i)|^{\frac{1}{2}}\};$$

$$\sum_{i=1}^n |(x, e_i)|^2 \leq n^{\frac{1}{2l}} \|x\| \left[ \sum_{i=1}^n |(x, e_i)|^m \right]^{\frac{1}{m}}, \quad m > 1, \quad \frac{1}{m} + \frac{1}{l} = 1;$$

$$\sum_{i=1}^n |(x, e_i)|^2 \leq \|x\| \sum_{i=1}^n |(x, e_i)|.$$

Corollaries 186 – 189 will produce the following results which do not contain the Fourier coefficients in the right side of the inequality.

Indeed, if one chooses  $c_i = (x, y_i)$  in (4.29), then

$$\left( \sum_{i=1}^n |(x, y_i)|^2 \right)^2 \leq \|x\|^2 n^{\frac{1}{p} + \frac{1}{t} - 1} \sum_{i=1}^n |(x, y_i)|^2 \left[ \sum_{i=1}^n \left( \sum_{j=1}^n |(y_i, y_j)|^q \right)^{\frac{u}{q}} \right]^{\frac{1}{u}},$$

giving the following Bombieri type inequality:

$$\sum_{i=1}^n |(x, y_i)|^2 \leq n^{\frac{1}{p} + \frac{1}{t} - 1} \|x\|^2 \left[ \sum_{i=1}^n \left( \sum_{j=1}^n |(y_i, y_j)|^q \right)^{\frac{u}{q}} \right]^{\frac{1}{u}},$$

where  $1 < p \leq 2$ ,  $1 < t \leq 2$ ,  $\frac{1}{p} + \frac{1}{q} = 1$ ,  $\frac{1}{t} + \frac{1}{u} = 1$ .

If in this inequality we consider  $p = q = t = u = 2$ , then

$$\sum_{i=1}^n |(x, y_i)|^2 \leq \|x\|^2 \left( \sum_{i,j=1}^n |(y_i, y_j)|^2 \right)^{\frac{1}{2}}.$$

For a different proof of this result see also [7].

In a similar way, if  $c_i = \overline{(x, y_i)}$  in (4.30), then

$$\sum_{i=1}^n |(x, y_i)|^2 \leq n^{\frac{1}{m}} \|x\|^2 \left( \sum_{i=1}^n \left[ \max_{1 \leq j \leq n} |(y_i, y_j)| \right]^t \right)^{\frac{1}{t}},$$

where  $m > 1$ ,  $\frac{1}{m} + \frac{1}{t} = 1$ .

Finally, if  $c_i = \overline{(x, y_i)}$  ( $i = 1, \dots, n$ ), is taken in (4.31), then

$$\sum_{i=1}^n |(x, y_i)|^2 \leq n \|x\|^2 \max_{1 \leq i, j \leq n} |(y_i, y_j)|.$$

**Remark 191** *Let us compare Bombieri's result*

$$\sum_{i=1}^n |(x, y_i)|^2 \leq \|x\|^2 \max_{1 \leq i \leq n} \left\{ \sum_{j=1}^n |(y_i, y_j)| \right\} \quad (4.41)$$

*with our result*

$$\sum_{i=1}^n |(x, y_i)|^2 \leq \|x\|^2 \left\{ \sum_{i,j=1}^n |(y_i, y_j)|^2 \right\}^{\frac{1}{2}}. \quad (4.42)$$

*Denote*

$$M_1 := \max_{1 \leq i \leq n} \left\{ \sum_{j=1}^n |(y_i, y_j)| \right\}$$

*and*

$$M_2 := \left[ \sum_{i,j=1}^n |(y_i, y_j)|^2 \right]^{\frac{1}{2}}.$$

If we choose the inner product space  $H = \mathbb{R}$ ,  $(x, y) := xy$  and  $n = 2$ , then for  $y_1 = a$ ,  $y_2 = b$ ,  $a, b > 0$ , we have

$$M_1 = \max \{a^2 + ab, ab + b^2\} = (a + b) \max(a, b),$$

$$M_2 = (a^4 + a^2b^2 + a^2b^2 + b^4)^{\frac{1}{2}} = a^2 + b^2.$$

Assume that  $a \geq b$ . Then  $M_1 = a^2 + ab \geq a^2 + b^2 = M_2$ , showing that, in this case, the bound provided by (4.42) is better than the bound provided by (4.41). If  $(y_i)_{1 \leq i \leq n}$  are orthonormal vectors, then  $M_1 = 1$ ,  $M_2 = \sqrt{n}$ , showing that in this case the Bombieri inequality (which becomes Bessel's inequality) provides a better bound than (4.42).

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## 4.3 Pečarić Type Inequalities

### 4.3.1 Introduction

In 1992, J.E. Pečarić [4] proved the following inequality for vectors in complex inner product spaces  $(H; (\cdot, \cdot))$ .

**Theorem 192** *Suppose that  $x, y_1, \dots, y_n$  are vectors in  $H$  and  $c_1, \dots, c_n$  are complex numbers. Then the following inequalities*

$$\begin{aligned} \left| \sum_{i=1}^n c_i (x, y_i) \right|^2 &\leq \|x\|^2 \sum_{i=1}^n |c_i|^2 \left( \sum_{j=1}^n |(y_i, y_j)| \right) \\ &\leq \|x\|^2 \sum_{i=1}^n |c_i|^2 \max_{1 \leq i \leq n} \left( \sum_{j=1}^n |(y_i, y_j)| \right), \end{aligned} \quad (4.43)$$

hold.

He also showed that for  $c_i = \overline{(x, y_i)}$ ,  $i \in \{1, \dots, n\}$ , one gets

$$\begin{aligned} \left( \sum_{i=1}^n |(x, y_i)|^2 \right)^2 &\leq \|x\|^2 \sum_{i=1}^n |(x, y_i)|^2 \left( \sum_{j=1}^n |(y_i, y_j)| \right) \\ &\leq \|x\|^2 \sum_{i=1}^n |(x, y_i)|^2 \max_{1 \leq i \leq n} \left( \sum_{j=1}^n |(y_i, y_j)| \right), \end{aligned}$$

which improves Bombieri's result [1] (see also [3, p. 394])

$$\sum_{i=1}^n |(x, y_i)|^2 \leq \|x\|^2 \max_{1 \leq i \leq n} \left( \sum_{j=1}^n |(y_i, y_j)| \right). \quad (4.44)$$

Note that (4.44) is in its turn a natural generalisation of *Bessel's inequality*

$$\sum_{i=1}^n |(x, e_i)|^2 \leq \|x\|^2, \quad x \in H, \quad (4.45)$$

which holds for the orthonormal vectors  $(e_i)_{1 \leq i \leq n}$ .

In this section, by following [2], we point out other related results to Pečarić's inequality (4.43) than the ones stated in the previous sections. Some results of Bombieri type are also mentioned.

### 4.3.2 Some Norm Inequalities

We start with the following lemma that is interesting in its own right [2].

**Lemma 193** Let  $z_1, \dots, z_n \in H$  and  $\alpha_1, \dots, \alpha_n \in \mathbb{K}$ . Then one has the inequalities:

$$\left\| \sum_{i=1}^n \alpha_i z_i \right\|^2 \leq \left( \sum_{i=1}^n |\alpha_i|^p \left( \sum_{j=1}^n |(z_i, z_j)| \right) \right)^{\frac{1}{p}} \times \left( \sum_{i=1}^n |\alpha_i|^q \left( \sum_{j=1}^n |(z_i, z_j)| \right) \right)^{\frac{1}{q}} \leq \begin{cases} \frac{A}{B} \\ C \end{cases}, \quad (4.46)$$

where

$$A := \begin{cases} \max_{1 \leq i \leq n} |\alpha_i|^2 \sum_{i,j=1}^n |(z_i, z_j)|; \\ \max_{1 \leq i \leq n} |\alpha_i| (\sum_{i=1}^n |\alpha_i|^{\gamma q})^{\frac{1}{\gamma q}} \left( \sum_{i,j=1}^n |(z_i, z_j)| \right)^{\frac{1}{p}} \left( \sum_{i=1}^n \left( \sum_{j=1}^n |(z_i, z_j)| \right)^{\delta} \right)^{\frac{1}{\delta q}}, \\ \quad \text{if } \gamma > 1, \frac{1}{\gamma} + \frac{1}{\delta} = 1; \\ \max_{1 \leq i \leq n} |\alpha_i| (\sum_{i=1}^n |\alpha_i|^q)^{\frac{1}{q}} \left( \sum_{i,j=1}^n |(z_i, z_j)| \right)^{\frac{1}{p}} \max_{1 \leq i \leq n} \left( \sum_{j=1}^n |(z_i, z_j)| \right)^{\frac{1}{q}}; \\ \max_{1 \leq i \leq n} |\alpha_i| (\sum_{i=1}^n |\alpha_i|^{\alpha p})^{\frac{1}{\alpha p}} \left( \sum_{i,j=1}^n |(z_i, z_j)| \right)^{\frac{1}{q}} \left( \sum_{i=1}^n \left( \sum_{j=1}^n |(z_i, z_j)| \right)^{\beta} \right)^{\frac{1}{\beta q}}, \\ \quad \text{if } \alpha > 1, \frac{1}{\alpha} + \frac{1}{\beta} = 1; \end{cases}$$

$$B := \begin{cases} (\sum_{i=1}^n |\alpha_i|^{\alpha p})^{\frac{1}{\alpha p}} (\sum_{i=1}^n |\alpha_i|^{\gamma q})^{\frac{1}{\gamma q}} \\ \quad \times \left( \sum_{i=1}^n \left( \sum_{j=1}^n |(z_i, z_j)| \right)^{\beta} \right)^{\frac{1}{p\beta}} \left( \sum_{i=1}^n \left( \sum_{j=1}^n |(z_i, z_j)| \right)^{\delta} \right)^{\frac{1}{\delta q}} \\ \quad \text{if } \alpha > 1, \frac{1}{\alpha} + \frac{1}{\beta} = 1 \text{ and } \gamma > 1, \frac{1}{\gamma} + \frac{1}{\delta} = 1; \\ (\sum_{i=1}^n |\alpha_i|^q)^{\frac{1}{q}} (\sum_{i=1}^n |\alpha_i|^{\alpha p})^{\frac{1}{\alpha p}} \max_{1 \leq i \leq n} \left( \sum_{j=1}^n |(z_i, z_j)| \right)^{\frac{1}{q}} \\ \quad \times \left( \sum_{i=1}^n \left( \sum_{j=1}^n |(z_i, z_j)| \right)^{\beta} \right)^{\frac{1}{p\beta}}, \quad \text{if } \alpha > 1, \frac{1}{\alpha} + \frac{1}{\beta} = 1; \end{cases}$$

and

$$C := \begin{cases} \max_{1 \leq i \leq n} |\alpha_i| (\sum_{i=1}^n |\alpha_i|^p)^{\frac{1}{p}} \max_{1 \leq i \leq n} \left( \sum_{j=1}^n |(z_i, z_j)| \right)^{\frac{1}{p}} \left( \sum_{i,j=1}^n |(z_i, z_j)| \right)^{\frac{1}{q}}; \\ (\sum_{i=1}^n |\alpha_i|^p)^{\frac{1}{p}} (\sum_{i=1}^n |\alpha_i|^{\gamma q})^{\frac{1}{\gamma q}} \max_{1 \leq i \leq n} \left( \sum_{j=1}^n |(z_i, z_j)| \right)^{\frac{1}{p}} \\ \quad \times \left( \sum_{i=1}^n \left( \sum_{j=1}^n |(z_i, z_j)| \right)^{\delta} \right)^{\frac{1}{\delta q}}, \quad \text{if } \gamma > 1, \frac{1}{\gamma} + \frac{1}{\delta} = 1; \\ (\sum_{i=1}^n |\alpha_i|^p)^{\frac{1}{p}} (\sum_{i=1}^n |\alpha_i|^q)^{\frac{1}{q}} \max_{1 \leq i \leq n} \left( \sum_{j=1}^n |(z_i, z_j)| \right), \end{cases}$$

where  $p > 1, \frac{1}{p} + \frac{1}{q} = 1$ .



**Proof.** We observe that

$$\begin{aligned}
\left\| \sum_{i=1}^n \alpha_i z_i \right\|^2 &= \left( \sum_{i=1}^n \alpha_i z_i, \sum_{j=1}^n \alpha_j z_j \right) \\
&= \sum_{i=1}^n \sum_{j=1}^n \alpha_i \overline{\alpha_j} (z_i, z_j) = \left| \sum_{i=1}^n \sum_{j=1}^n \alpha_i \overline{\alpha_j} (z_i, z_j) \right| \\
&\leq \sum_{i=1}^n \sum_{j=1}^n |\alpha_i| |\alpha_j| |(z_i, z_j)| =: M.
\end{aligned} \tag{4.47}$$

If one uses the Hölder inequality for double sums, i.e., we recall it

$$\sum_{i,j=1}^n m_{ij} a_{ij} b_{ij} \leq \left( \sum_{i,j=1}^n m_{ij} a_{ij}^p \right)^{\frac{1}{p}} \left( \sum_{i,j=1}^n m_{ij} b_{ij}^q \right)^{\frac{1}{q}},$$

where  $m_{ij}, a_{ij}, b_{ij} \geq 0$ ,  $\frac{1}{p} + \frac{1}{q} = 1$ ,  $p > 1$ ; then

$$\begin{aligned}
M &\leq \left( \sum_{i,j=1}^n |(z_i, z_j)| |\alpha_i|^p \right)^{\frac{1}{p}} \left( \sum_{i,j=1}^n |(z_i, z_j)| |\alpha_i|^q \right)^{\frac{1}{q}} \\
&= \left( \sum_{i=1}^n |\alpha_i|^p \left( \sum_{j=1}^n |(z_i, z_j)| \right) \right)^{\frac{1}{p}} \left( \sum_{i=1}^n |\alpha_i|^q \left( \sum_{j=1}^n |(z_i, z_j)| \right) \right)^{\frac{1}{q}},
\end{aligned} \tag{4.48}$$

and the first inequality in (4.46) is proved.

Observe that

$$\sum_{i=1}^n |\alpha_i|^p \left( \sum_{j=1}^n |(z_i, z_j)| \right) \leq \begin{cases} \max_{1 \leq i \leq n} |\alpha_i|^p \sum_{i,j=1}^n |(z_i, z_j)|; \\ (\sum_{i=1}^n |\alpha_i|^{\alpha p})^{\frac{1}{\alpha}} \left( \sum_{i=1}^n \left( \sum_{j=1}^n |(z_i, z_j)| \right)^{\beta} \right)^{\frac{1}{\beta}} \\ \quad \text{if } \alpha > 1, \frac{1}{\alpha} + \frac{1}{\beta} = 1; \\ \sum_{i=1}^n |\alpha_i|^p \max_{1 \leq i \leq n} \left( \sum_{j=1}^n |(z_i, z_j)| \right); \end{cases}$$

giving

$$\left( \sum_{i=1}^n |\alpha_i|^p \left( \sum_{j=1}^n |(z_i, z_j)| \right) \right)^{\frac{1}{p}} \leq \begin{cases} \max_{1 \leq i \leq n} |\alpha_i| \left( \sum_{j=1}^n |(z_i, z_j)| \right)^{\frac{1}{p}}; \\ (\sum_{i=1}^n |\alpha_i|^{\alpha p})^{\frac{1}{\alpha p}} \left( \sum_{i=1}^n \left( \sum_{j=1}^n |(z_i, z_j)| \right)^{\beta} \right)^{\frac{1}{\beta p}} \\ \quad \text{if } \alpha > 1, \frac{1}{\alpha} + \frac{1}{\beta} = 1; \\ (\sum_{i=1}^n |\alpha_i|^p)^{\frac{1}{p}} \max_{1 \leq i \leq n} \left( \sum_{j=1}^n |(z_i, z_j)| \right)^{\frac{1}{p}}. \end{cases} \quad (4.49)$$

Similarly, we have

$$\left( \sum_{i=1}^n |\alpha_i|^q \left( \sum_{j=1}^n |(z_i, z_j)| \right) \right)^{\frac{1}{q}} \leq \begin{cases} \max_{1 \leq i \leq n} |\alpha_i| \left( \sum_{j=1}^n |(z_i, z_j)| \right)^{\frac{1}{q}} \\ (\sum_{i=1}^n |\alpha_i|^{\gamma q})^{\frac{1}{\gamma q}} \left( \sum_{i=1}^n \left( \sum_{j=1}^n |(z_i, z_j)| \right)^{\delta} \right)^{\frac{1}{\delta q}} \\ \quad \text{if } \gamma > 1, \frac{1}{\gamma} + \frac{1}{\delta} = 1; \\ (\sum_{i=1}^n |\alpha_i|^q)^{\frac{1}{q}} \max_{1 \leq i \leq n} \left( \sum_{j=1}^n |(z_i, z_j)| \right)^{\frac{1}{q}}. \end{cases} \quad (4.50)$$

Using (4.47) and (4.49) – (4.50), we deduce the 9 inequalities in the second part of (4.46). ■

If we choose  $p = q = 2$ , then the following result holds [2].

**Corollary 194** *If  $z_1, \dots, z_n \in H$  and  $\alpha_1, \dots, \alpha_n \in \mathbb{K}$ , then one has*

$$\left\| \sum_{i=1}^n \alpha_i z_i \right\|^2 \leq \sum_{i=1}^n |\alpha_i|^2 \left( \sum_{j=1}^n |(z_i, z_j)| \right) \leq \begin{cases} D \\ E \\ F \end{cases}, \quad (4.51)$$

where

$$D := \begin{cases} \max_{1 \leq i \leq n} |\alpha_i|^2 \sum_{i,j=1}^n |(z_i, z_j)|; \\ \max_{1 \leq i \leq n} |\alpha_i| \left( \sum_{i=1}^n |\alpha_i|^{2\gamma} \right)^{\frac{1}{2\gamma}} \left( \sum_{i,j=1}^n |(z_i, z_j)| \right)^{\frac{1}{2}} \left( \sum_{i=1}^n \left( \sum_{j=1}^n |(z_i, z_j)| \right)^\delta \right)^{\frac{1}{2\delta}}, \\ \quad \text{if } \gamma > 1, \frac{1}{\gamma} + \frac{1}{\delta} = 1; \\ \max_{1 \leq i \leq n} |\alpha_i| \left( \sum_{i=1}^n |\alpha_i|^2 \right)^{\frac{1}{2}} \left( \sum_{i,j=1}^n |(z_i, z_j)| \right)^{\frac{1}{2}} \max_{1 \leq i \leq n} \left( \sum_{j=1}^n |(z_i, z_j)| \right)^{\frac{1}{2}}; \end{cases}$$

$$E := \begin{cases} \max_{1 \leq i \leq n} |\alpha_i| \left( \sum_{i=1}^n |\alpha_i|^{2\alpha} \right)^{\frac{1}{2\alpha}} \left( \sum_{i,j=1}^n |(z_i, z_j)| \right)^{\frac{1}{2}} \left( \sum_{i=1}^n \left( \sum_{j=1}^n |(z_i, z_j)| \right)^\beta \right)^{\frac{1}{2\beta}}, \\ \quad \text{if } \alpha > 1, \frac{1}{\alpha} + \frac{1}{\beta} = 1; \\ \left( \sum_{i=1}^n |\alpha_i|^{2\alpha} \right)^{\frac{1}{2\alpha}} \left( \sum_{i=1}^n |\alpha_i|^{2\gamma} \right)^{\frac{1}{2\gamma}} \left( \sum_{i=1}^n \left( \sum_{j=1}^n |(z_i, z_j)| \right)^\beta \right)^{\frac{1}{2\beta}} \\ \quad \times \left( \sum_{i=1}^n \left( \sum_{j=1}^n |(z_i, z_j)| \right)^\delta \right)^{\frac{1}{2\delta}} \quad \text{if } \alpha > 1, \frac{1}{\alpha} + \frac{1}{\beta} = 1 \text{ and } \gamma > 1, \frac{1}{\gamma} + \frac{1}{\delta} = 1; \\ \left( \sum_{i=1}^n |\alpha_i|^2 \right)^{\frac{1}{2}} \left( \sum_{i=1}^n |\alpha_i|^{2\alpha} \right)^{\frac{1}{2\alpha}} \max_{1 \leq i \leq n} \left( \sum_{j=1}^n |(z_i, z_j)| \right)^{\frac{1}{2}} \left( \sum_{i=1}^n \left( \sum_{j=1}^n |(z_i, z_j)| \right)^\beta \right)^{\frac{1}{2\beta}}, \\ \quad \text{if } \alpha > 1, \frac{1}{\alpha} + \frac{1}{\beta} = 1; \end{cases}$$

and

$$F := \begin{cases} \max_{1 \leq i \leq n} |\alpha_i| \left( \sum_{i=1}^n |\alpha_i|^2 \right)^{\frac{1}{2}} \max_{1 \leq i \leq n} \left( \sum_{j=1}^n |(z_i, z_j)| \right)^{\frac{1}{2}} \left( \sum_{i,j=1}^n |(z_i, z_j)| \right)^{\frac{1}{2}}; \\ \left( \sum_{i=1}^n |\alpha_i|^2 \right)^{\frac{1}{2}} \left( \sum_{i=1}^n |\alpha_i|^{2\gamma} \right)^{\frac{1}{2\gamma}} \max_{1 \leq i \leq n} \left( \sum_{j=1}^n |(z_i, z_j)| \right)^{\frac{1}{2}} \left( \sum_{i=1}^n \left( \sum_{j=1}^n |(z_i, z_j)| \right)^\delta \right)^{\frac{1}{2\delta}}, \\ \quad \text{if } \gamma > 1, \frac{1}{\gamma} + \frac{1}{\delta} = 1; \\ \sum_{i=1}^n |\alpha_i|^2 \max_{1 \leq i \leq n} \left( \sum_{j=1}^n |(z_i, z_j)| \right). \end{cases}$$

### 4.3.3 Some Pečarić Type Inequalities

We are now able to point out the following result obtained in [2], which complements and generalises the inequality (4.43) due to J. Pečarić.

**Theorem 195** *Let  $x, y_1, \dots, y_n$  be vectors of an inner product space  $(H; (\cdot, \cdot))$  and  $c_1, \dots, c_n \in \mathbb{K}$ . Then one has the inequalities:*

$$\left| \sum_{i=1}^n c_i (x, y_i) \right|^2 \leq \|x\|^2 \left( \sum_{i=1}^n |c_i|^p \left( \sum_{j=1}^n |(y_i, y_j)| \right) \right)^{\frac{1}{p}} \left( \sum_{i=1}^n |c_i|^q \left( \sum_{j=1}^n |(y_i, y_j)| \right) \right)^{\frac{1}{q}} \leq \|x\|^2 \times \begin{cases} G \\ H \\ I \end{cases}, \quad (4.52)$$

where

$$G := \begin{cases} \max_{1 \leq i \leq n} |c_i|^2 \sum_{j=1}^n |(y_i, y_j)|; \\ \max_{1 \leq i \leq n} |c_i| \left( \sum_{i=1}^n |c_i|^{\gamma q} \right)^{\frac{1}{\gamma q}} \left( \sum_{i,j=1}^n |(y_i, y_j)| \right)^{\frac{1}{p}} \\ \quad \times \left( \sum_{i=1}^n \left( \sum_{j=1}^n |(y_i, y_j)| \right)^{\delta} \right)^{\frac{1}{\delta q}}, \quad \text{if } \gamma > 1, \frac{1}{\gamma} + \frac{1}{\delta} = 1; \\ \max_{1 \leq i \leq n} |c_i| \left( \sum_{i=1}^n |c_i|^q \right)^{\frac{1}{q}} \left( \sum_{i,j=1}^n |(y_i, y_j)| \right)^{\frac{1}{p}} \max_{1 \leq i \leq n} \left( \sum_{j=1}^n |(y_i, y_j)| \right)^{\frac{1}{q}}; \end{cases}$$

$$H := \begin{cases} \max_{1 \leq i \leq n} |c_i| \left( \sum_{i=1}^n |c_i|^{\alpha p} \right)^{\frac{1}{\alpha p}} \left( \sum_{i,j=1}^n |(y_i, y_j)| \right)^{\frac{1}{q}} \\ \quad \times \left( \sum_{i=1}^n \left( \sum_{j=1}^n |(y_i, y_j)| \right)^{\beta} \right)^{\frac{1}{p\beta}}, \quad \text{if } \alpha > 1, \frac{1}{\alpha} + \frac{1}{\beta} = 1; \\ \left( \sum_{i=1}^n |c_i|^{\alpha p} \right)^{\frac{1}{\alpha p}} \left( \sum_{i=1}^n |c_i|^{\gamma q} \right)^{\frac{1}{\gamma q}} \left( \sum_{i=1}^n \left( \sum_{j=1}^n |(y_i, y_j)| \right)^{\beta} \right)^{\frac{1}{p\beta}} \\ \quad \times \left( \sum_{i=1}^n \left( \sum_{j=1}^n |(y_i, y_j)| \right)^{\delta} \right)^{\frac{1}{\delta q}} \\ \quad \text{if } \alpha > 1, \frac{1}{\alpha} + \frac{1}{\beta} = 1 \text{ and } \gamma > 1, \frac{1}{\gamma} + \frac{1}{\delta} = 1; \\ \left( \sum_{i=1}^n |c_i|^q \right)^{\frac{1}{q}} \left( \sum_{i=1}^n |c_i|^{\alpha p} \right)^{\frac{1}{\alpha p}} \max_{1 \leq i \leq n} \left( \sum_{j=1}^n |(y_i, y_j)| \right)^{\frac{1}{q}} \\ \quad \times \left( \sum_{i=1}^n \left( \sum_{j=1}^n |(y_i, y_j)| \right)^{\beta} \right)^{\frac{1}{p\beta}}, \quad \text{if } \alpha > 1, \frac{1}{\alpha} + \frac{1}{\beta} = 1; \end{cases}$$

and

$$I := \begin{cases} \max_{1 \leq i \leq n} |c_i| \left( \sum_{i=1}^n |c_i|^p \right)^{\frac{1}{p}} \max_{1 \leq i \leq n} \left( \sum_{j=1}^n |(y_i, y_j)| \right)^{\frac{1}{p}} \left( \sum_{i,j=1}^n |(y_i, y_j)| \right)^{\frac{1}{q}} ; \\ \left( \sum_{i=1}^n |c_i|^p \right)^{\frac{1}{p}} \left( \sum_{i=1}^n |c_i|^{\gamma q} \right)^{\frac{1}{\gamma q}} \max_{1 \leq i \leq n} \left( \sum_{j=1}^n |(y_i, y_j)| \right)^{\frac{1}{p}} \\ \quad \times \left( \sum_{i=1}^n \left( \sum_{j=1}^n |(y_i, y_j)| \right)^{\delta} \right)^{\frac{1}{\delta q}}, \quad \text{if } \gamma > 1, \frac{1}{\gamma} + \frac{1}{\delta} = 1; \\ \left( \sum_{i=1}^n |c_i|^p \right)^{\frac{1}{p}} \left( \sum_{i=1}^n |c_i|^q \right)^{\frac{1}{q}} \max_{1 \leq i \leq n} \left( \sum_{j=1}^n |(y_i, y_j)| \right); \end{cases}$$

for  $p > 1$ ,  $\frac{1}{p} + \frac{1}{q} = 1$ .

**Proof.** We note that

$$\sum_{i=1}^n c_i(x, y_i) = \left( x, \sum_{i=1}^n \overline{c_i} y_i \right).$$

Using Schwarz's inequality in inner product spaces, we have

$$\left| \sum_{i=1}^n c_i(x, y_i) \right|^2 \leq \|x\|^2 \left\| \sum_{i=1}^n \overline{c_i} y_i \right\|^2.$$

Finally, using Lemma 193 with  $\alpha_i = \overline{c_i}$ ,  $z_i = y_i$  ( $i = 1, \dots, n$ ), we deduce the desired inequality (4.52). ■

**Remark 196** If in (4.52) we choose  $p = q = 2$ , we obtain amongst others, the result (4.43) due to J. Pečarić.

#### 4.3.4 More Results of Bombieri Type

The following results of Bombieri type hold [2].

**Theorem 197** Let  $x, y_1, \dots, y_n \in H$ . Then one has the inequality:

$$\begin{aligned} & \sum_{i=1}^n |(x, y_i)|^2 \tag{4.53} \\ & \leq \|x\| \left[ \sum_{i=1}^n |(x, y_i)|^p \left( \sum_{j=1}^n |(y_i, y_j)| \right) \right]^{\frac{1}{2p}} \left[ \sum_{i=1}^n |(x, y_i)|^q \left( \sum_{j=1}^n |(y_i, y_j)| \right) \right]^{\frac{1}{2q}} \\ & \leq \|x\| \times \begin{cases} J \\ K \\ L \end{cases}, \end{aligned}$$

where

$$J := \begin{cases} \max_{1 \leq i \leq n} |(x, y_i)| \left( \sum_{i,j=1}^n |(y_i, y_j)| \right)^{\frac{1}{2}}; \\ \max_{1 \leq i \leq n} |(x, y_i)|^{\frac{1}{2}} \left( \sum_{i=1}^n |(x, y_i)|^{\gamma q} \right)^{\frac{1}{2\gamma q}} \left( \sum_{i,j=1}^n |(y_i, y_j)| \right)^{\frac{1}{2p}} \\ \quad \times \left( \sum_{i=1}^n \left( \sum_{j=1}^n |(y_i, y_j)| \right)^{\delta} \right)^{\frac{1}{2\delta q}}, & \text{if } \gamma > 1, \frac{1}{\gamma} + \frac{1}{\delta} = 1; \\ \max_{1 \leq i \leq n} |(x, y_i)|^{\frac{1}{2}} \left( \sum_{i=1}^n |(x, y_i)|^q \right)^{\frac{1}{2q}} \left( \sum_{i,j=1}^n |(y_i, y_j)| \right)^{\frac{1}{2p}} \max_{1 \leq i \leq n} \left( \sum_{j=1}^n |(y_i, y_j)| \right)^{\frac{1}{2q}}; \end{cases}$$

$$K := \begin{cases} \max_{1 \leq i \leq n} |(x, y_i)|^{\frac{1}{2}} \left( \sum_{i=1}^n |(x, y_i)|^{\alpha p} \right)^{\frac{1}{2\alpha\beta}} \left( \sum_{i,j=1}^n |(y_i, y_j)| \right)^{\frac{1}{2q}} \\ \quad \times \left( \sum_{i=1}^n \left( \sum_{j=1}^n |(y_i, y_j)| \right)^{\beta} \right)^{\frac{1}{p\beta}}, & \text{if } \alpha > 1, \frac{1}{\alpha} + \frac{1}{\beta} = 1; \\ \left( \sum_{i=1}^n |(x, y_i)|^{\alpha p} \right)^{\frac{1}{2\alpha p}} \left( \sum_{i=1}^n |(x, y_i)|^{\gamma q} \right)^{\frac{1}{2\gamma q}} \left( \sum_{i=1}^n \left( \sum_{j=1}^n |(y_i, y_j)| \right)^{\beta} \right)^{\frac{1}{2p\beta}} \\ \quad \times \left( \sum_{i=1}^n \left( \sum_{j=1}^n |(y_i, y_j)| \right)^{\delta} \right)^{\frac{1}{2\delta q}} & \text{if } \alpha > 1, \frac{1}{\alpha} + \frac{1}{\beta} = 1 \\ & \text{and } \gamma > 1, \frac{1}{\gamma} + \frac{1}{\delta} = 1; \\ \left( \sum_{i=1}^n |(x, y_i)|^q \right)^{\frac{1}{2q}} \left( \sum_{i=1}^n |(x, y_i)|^{\alpha p} \right)^{\frac{1}{2\alpha p}} \max_{1 \leq i \leq n} \left( \sum_{j=1}^n |(y_i, y_j)| \right)^{\frac{1}{2p}} \\ \quad \times \left( \sum_{i=1}^n \left( \sum_{j=1}^n |(y_i, y_j)| \right)^{\beta} \right)^{\frac{1}{2p\beta}}, & \text{if } \alpha > 1, \frac{1}{\alpha} + \frac{1}{\beta} = 1; \end{cases}$$

and

$$L := \begin{cases} \max_{1 \leq i \leq n} |(x, y_i)|^{\frac{1}{2}} \left( \sum_{i=1}^n |(x, y_i)|^p \right)^{\frac{1}{2p}} \max_{1 \leq i \leq n} \left( \sum_{j=1}^n |(y_i, y_j)| \right)^{\frac{1}{2p}} \\ \quad \times \left( \sum_{i,j=1}^n |(y_i, y_j)| \right)^{\frac{1}{2q}}; \\ \left( \sum_{i=1}^n |(x, y_i)|^p \right)^{\frac{1}{2p}} \left( \sum_{i=1}^n |(x, y_i)|^{\gamma q} \right)^{\frac{1}{2\gamma q}} \max_{1 \leq i \leq n} \left( \sum_{j=1}^n |(y_i, y_j)| \right)^{\frac{1}{2p}} \\ \quad \times \left( \sum_{i=1}^n \left( \sum_{j=1}^n |(y_i, y_j)| \right)^{\delta} \right)^{\frac{1}{2\delta q}}, & \text{if } \gamma > 1, \frac{1}{\gamma} + \frac{1}{\delta} = 1; \\ \left( \sum_{i=1}^n |(x, y_i)|^p \right)^{\frac{1}{2p}} \left( \sum_{i=1}^n |(x, y_i)|^q \right)^{\frac{1}{2q}} \max_{1 \leq i \leq n} \left( \sum_{j=1}^n |(y_i, y_j)| \right)^{\frac{1}{2}}, \end{cases}$$

for  $p > 1, \frac{1}{p} + \frac{1}{q} = 1$ .

**Proof.** The proof follows by Theorem 195 on choosing  $c_i = \overline{(x, y_i)}$ ,  $i \in \{1, \dots, n\}$  and taking the square root in both sides of the inequalities involved. We omit the details. ■

**Remark 198** *We observe, by the last inequality in (4.53), that*

$$\frac{\left(\sum_{i=1}^n |(x, y_i)|^2\right)^2}{\left(\sum_{i=1}^n |(x, y_i)|^p\right)^{\frac{1}{p}} \left(\sum_{i=1}^n |(x, y_i)|^q\right)^{\frac{1}{q}}} \leq \|x\|^2 \max_{1 \leq i \leq n} \left(\sum_{j=1}^n |(y_i, y_j)|\right),$$

where  $p > 1$ ,  $\frac{1}{p} + \frac{1}{q} = 1$ .

*If in this inequality we choose  $p = q = 2$ , then we recapture Bombieri's result (4.44).*

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## Chapter 5

# Some Grüss' Type Inequalities for $n$ -Tuples of Vectors

### 5.1 Introduction

We start by recalling some of the most important Grüss type discrete inequalities for real numbers that are available in the literature.

1. (1950) *Biernacki, Pidek, Ryll-Nardzewski* [1].

If  $\bar{\mathbf{a}} = (a_1, \dots, a_n)$  and  $\bar{\mathbf{b}} = (b_1, \dots, b_n)$  are  $n$ -tuples of real numbers such that there exists the real numbers  $a, A, b, B$  with

$$a \leq a_i \leq A, \quad b \leq b_i \leq B, \quad i \in \{1, \dots, n\}, \quad (5.1)$$

then

$$\begin{aligned} |C_n(\bar{\mathbf{a}}, \bar{\mathbf{b}})| &\leq \frac{1}{n} \left[ \frac{n}{2} \right] \left( 1 - \frac{1}{n} \left[ \frac{n}{2} \right] \right) (A - a)(B - b) \\ &= \frac{1}{n^2} \left[ \frac{n^4}{4} \right] (A - a)(B - b) \\ &\leq \frac{1}{4} (A - a)(B - b). \end{aligned}$$

2. (1988) *Andrica-Badea* [2].

Let  $\bar{\mathbf{a}}, \bar{\mathbf{b}}$  satisfy (5.1) and  $\bar{\mathbf{p}} = (p_1, \dots, p_n)$  be an  $n$ -tuple of nonnegative numbers with  $P_n > 0$ . If  $S$  is a subset of  $\{1, \dots, n\}$  that minimises the

expression

$$\left| \sum_{i \in S} q_i - \frac{1}{2} Q_n \right|,$$

then

$$\begin{aligned} C_n(\bar{\mathbf{p}}, \bar{\mathbf{a}}, \bar{\mathbf{b}}) &\leq \frac{Q_S}{Q_n} \left( 1 - \frac{Q_S}{Q_n} \right) (A - a)(B - b) \\ &\leq \frac{1}{4} (A - a)(B - b), \end{aligned}$$

where  $Q_S := \sum_{i \in S} Q_i$ .

3. (2000) *Dragomir-Booth* [3].

If  $\bar{\mathbf{a}}, \bar{\mathbf{b}}$  are real  $n$ -tuples and  $\bar{\mathbf{p}}$  is nonnegative with  $P_n > 0$ , then

$$|C_n(\bar{\mathbf{p}}, \bar{\mathbf{a}}, \bar{\mathbf{b}})| \leq \max_{1 \leq j \leq n-1} |\Delta a_j| \max_{1 \leq j \leq n-1} |\Delta b_j| C_n(\bar{\mathbf{p}}, \bar{\mathbf{e}}, \bar{\mathbf{e}}),$$

where  $\bar{\mathbf{e}} = (1, 2, \dots, n)$  and  $\Delta a_j := a_{j+1} - a_j$  is the forward difference,  $j = 1, \dots, n-1$ . Note that

$$C_n(\bar{\mathbf{p}}, \bar{\mathbf{e}}, \bar{\mathbf{e}}) = \frac{1}{P_n^2} \sum_{i=1}^n i^2 p_i - \left( \frac{1}{P_n} \sum_{i=1}^n i p_i \right)^2.$$

In particular, we have

$$|C_n(\bar{\mathbf{a}}, \bar{\mathbf{b}})| \leq \frac{1}{12} (n^2 - 1) \max_{1 \leq j \leq n-1} |\Delta a_j| \max_{1 \leq j \leq n-1} |\Delta b_j|.$$

The constant  $\frac{1}{12}$  is best possible.

4. (2002) *Dragomir* [4].

With the assumptions in 3, the following inequality holds

$$|C_n(\bar{\mathbf{p}}, \bar{\mathbf{a}}, \bar{\mathbf{b}})| \leq \frac{1}{P_n^2} \sum_{1 \leq j < i \leq n} (i - j) \left( \sum_{k=1}^{n-1} |\Delta a_k|^p \right)^{\frac{1}{p}} \left( \sum_{k=1}^{n-1} |\Delta b_k|^q \right)^{\frac{1}{q}},$$

where  $p > 1$ ,  $\frac{1}{p} + \frac{1}{q} = 1$ .

In particular, we have

$$|C_n(\bar{\mathbf{a}}, \bar{\mathbf{b}})| \leq \frac{1}{6} \cdot \frac{n^2 - 1}{n} \left( \sum_{k=1}^{n-1} |\Delta a_k|^p \right)^{\frac{1}{p}} \left( \sum_{k=1}^{n-1} |\Delta b_k|^q \right)^{\frac{1}{q}}.$$

The constant  $\frac{1}{6}$  is best possible.

5. (2002) *Dragomir* [5].

The following inequality holds, where  $\bar{\mathbf{p}}, \bar{\mathbf{a}}, \bar{\mathbf{b}}$  and  $P_n$  are as in assumption 3,

$$|C_n(\bar{\mathbf{p}}, \bar{\mathbf{a}}, \bar{\mathbf{b}})| \leq \frac{1}{2} \cdot \frac{1}{P_n^2} \sum_{i=1}^n p_i (P_n - p_i) \sum_{k=1}^{n-1} |\Delta a_k| \sum_{k=1}^{n-1} |\Delta b_k|.$$

In particular, we have

$$|C_n(\bar{\mathbf{a}}, \bar{\mathbf{b}})| \leq \frac{1}{2} \left(1 - \frac{1}{n}\right) \sum_{k=1}^{n-1} |\Delta a_k| \sum_{k=1}^{n-1} |\Delta b_k|.$$

The constant  $\frac{1}{2}$  is sharp.

6. (2002) *Cerone-Dragomir* [6].

If  $\bar{\mathbf{a}}, \bar{\mathbf{b}}$  are real  $n$ -tuples and  $\bar{\mathbf{p}}$  is a positive  $n$ -tuple and there exists  $m, M \in \mathbb{R}$  such that

$$m \leq a_i \leq M,$$

then one has the inequality

$$|C_n(\bar{\mathbf{p}}, \bar{\mathbf{a}}, \bar{\mathbf{b}})| \leq \frac{1}{2} (M - m) \frac{1}{P_n} \sum_{i=1}^n p_i \left| b_i - \frac{1}{P_n} \sum_{j=1}^n p_j b_j \right|.$$

The constant  $\frac{1}{2}$  is best possible.

In particular, we have

$$|C_n(\bar{\mathbf{a}}, \bar{\mathbf{b}})| \leq \frac{1}{2} (M - m) \cdot \frac{1}{n} \sum_{i=1}^n \left| b_i - \frac{1}{n} \sum_{j=1}^n b_j \right|.$$

The constant  $\frac{1}{2}$  is best possible.

The main aim of this chapter is to present some extensions of the above results holding in the general setting of  $n$ -tuples of vectors in an inner product space.

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## 5.2 The Version for Norms

### 5.2.1 Preliminary Results

The following lemma is of interest in itself [1].

**Lemma 199** *Let  $(H; \langle \cdot, \cdot \rangle)$  be an inner product space over the real or complex number field  $\mathbb{K}$ ,  $x_i \in H$  and  $p_i \geq 0$  ( $i = 1, \dots, n$ ) such that  $\sum_{i=1}^n p_i = 1$  ( $n \geq 2$ ). If  $x, X \in H$  are such that*

$$\operatorname{Re} \langle X - x_i, x_i - x \rangle \geq 0 \text{ for all } i \in \{1, \dots, n\}, \quad (5.2)$$

or equivalently,

$$\left\| x_i - \frac{x + X}{2} \right\| \leq \frac{1}{2} \|X - x\| \text{ for all } i \in \{1, \dots, n\},$$

then we have the inequality

$$0 \leq \sum_{i=1}^n p_i \|x_i\|^2 - \left\| \sum_{i=1}^n p_i x_i \right\|^2 \leq \frac{1}{4} \|X - x\|^2. \quad (5.3)$$

The constant  $\frac{1}{4}$  is sharp.

**Proof.** Define

$$I_1 := \left\langle X - \sum_{i=1}^n p_i x_i, \sum_{i=1}^n p_i x_i - x \right\rangle$$

and

$$I_2 := \sum_{i=1}^n p_i \langle X - x_i, x_i - x \rangle.$$

Then

$$I_1 = \sum_{i=1}^n p_i \langle X, x_i \rangle - \langle X, x \rangle - \left\| \sum_{i=1}^n p_i x_i \right\|^2 + \sum_{i=1}^n p_i \langle x_i, x \rangle$$

and

$$I_2 = \sum_{i=1}^n p_i \langle X, x_i \rangle - \langle X, x \rangle - \sum_{i=1}^n p_i \|x_i\|^2 + \sum_{i=1}^n p_i \langle x_i, x \rangle.$$

Consequently,

$$I_1 - I_2 = \sum_{i=1}^n p_i \|x_i\|^2 - \left\| \sum_{i=1}^n p_i x_i \right\|^2. \quad (5.4)$$

Taking the real value in (5.4), we can state that

$$\begin{aligned} & \sum_{i=1}^n p_i \|x_i\|^2 - \left\| \sum_{i=1}^n p_i x_i \right\|^2 \\ &= \operatorname{Re} \left\langle X - \sum_{i=1}^n p_i x_i, \sum_{i=1}^n p_i x_i - x \right\rangle - \sum_{i=1}^n p_i \operatorname{Re} \langle X - x_i, x_i - x \rangle, \end{aligned} \quad (5.5)$$

which is also an identity of interest in itself.

Using the assumption (5.2), we can conclude, by (5.5), that

$$\sum_{i=1}^n p_i \|x_i\|^2 - \left\| \sum_{i=1}^n p_i x_i \right\|^2 \leq \operatorname{Re} \left\langle X - \sum_{i=1}^n p_i x_i, \sum_{i=1}^n p_i x_i - x \right\rangle. \quad (5.6)$$

It is known that if  $y, z \in H$ , then

$$4 \operatorname{Re} \langle z, y \rangle \leq \|z + y\|^2, \quad (5.7)$$

with equality iff  $z = y$ .

Now, by (5.7) we can state that

$$\begin{aligned} \operatorname{Re} \left\langle X - \sum_{i=1}^n p_i x_i, \sum_{i=1}^n p_i x_i - x \right\rangle &\leq \frac{1}{4} \left\| X - \sum_{i=1}^n p_i x_i + \sum_{i=1}^n p_i x_i - x \right\|^2 \\ &= \frac{1}{4} \|X - x\|^2. \end{aligned}$$

Using (5.6), we obtain (5.3).

To prove the sharpness of the constant  $\frac{1}{4}$ , let us assume that the inequality (5.3) holds with a constant  $c > 0$ , i.e.,

$$0 \leq \sum_{i=1}^n p_i \|x_i\|^2 - \left\| \sum_{i=1}^n p_i x_i \right\|^2 \leq c \|X - x\|^2 \quad (5.8)$$

for all  $p_i, x_i$  and  $n$  as in the hypothesis of Lemma 199.

Assume that  $n = 2$ ,  $p_1 = p_2 = \frac{1}{2}$ ,  $x_1 = x$  and  $x_2 = X$  with  $x, X \in H$  and  $x \neq X$ . Then, obviously,

$$\langle X - x_1, x_1 - x \rangle = \langle X - x_2, x_2 - x \rangle = 0,$$

which shows that the condition (5.2) holds.

If we replace  $n, p_1, p_2, x_1, x_2$  in (5.8), we obtain

$$\begin{aligned} \sum_{i=1}^2 p_i \|x_i\|^2 - \left\| \sum_{i=1}^2 p_i x_i \right\|^2 &= \frac{1}{2} (\|x\|^2 + \|X\|^2) - \left\| \frac{x + X}{2} \right\|^2 \\ &= \frac{1}{4} \|x - X\|^2 \\ &\leq c \|x - X\|^2, \end{aligned}$$

from where we deduce that  $c \geq \frac{1}{4}$ , which proves the sharpness of the constant  $\frac{1}{4}$ . ■

**Remark 200** *The assumption (5.2) can be replaced by the more general condition*

$$\sum_{i=1}^n p_i \operatorname{Re} \langle X - x_i, x_i - x \rangle \geq 0, \quad (5.9)$$

*and the conclusion (5.3) will still remain valid.*

The following corollary is natural.

**Corollary 201** *Let  $a_i \in \mathbb{K}$ ,  $p_i \geq 0$ ,  $(i = 1, \dots, n)$  ( $n \geq 2$ ) with  $\sum_{i=1}^n p_i = 1$ . If  $a, A \in \mathbb{K}$  are such that*

$$\operatorname{Re}[(A - a_i)(\bar{a}_i - \bar{a})] \geq 0 \text{ for all } i \in \{1, \dots, n\}, \quad (5.10)$$

*then we have the inequality*

$$0 \leq \sum_{i=1}^n p_i |a_i|^2 - \left| \sum_{i=1}^n p_i a_i \right|^2 \leq \frac{1}{4} |A - a|^2. \quad (5.11)$$

*The constant  $\frac{1}{4}$  is sharp.*

The proof follows by the above lemma by choosing  $H = \mathbb{K}$ ,  $\langle x, y \rangle := x\bar{y}$ ,  $x_i = a_i$ ,  $x = a$  and  $X = A$ . We omit the details.

**Remark 202** *The condition (5.10) can be replaced by the more general assumption*

$$\sum_{i=1}^n p_i \operatorname{Re}[(A - a_i)(\bar{a}_i - \bar{a})] \geq 0.$$

### 5.2.2 A Discrete Inequality of Grüss' Type

The following Grüss type inequality holds [1].

**Theorem 203** *Let  $(H; \langle \cdot, \cdot \rangle)$  be an inner product space over  $\mathbb{K}$ ,  $\mathbb{K} = \mathbb{R}, \mathbb{C}$ ,  $x_i \in H$ ,  $a_i \in \mathbb{K}$ ,  $p_i \geq 0$  ( $i = 1, \dots, n$ ) ( $n \geq 2$ ) with  $\sum_{i=1}^n p_i = 1$ . If  $a, A \in \mathbb{K}$  and  $x, X \in H$  are such that*

$$\operatorname{Re}[(A - a_i)(\bar{a}_i - \bar{a})] \geq 0, \quad \operatorname{Re} \langle X - x_i, x_i - x \rangle \geq 0 \quad (5.12)$$

*for all  $i \in \{1, \dots, n\}$ ; then we have the inequality*

$$0 \leq \left\| \sum_{i=1}^n p_i a_i x_i - \sum_{i=1}^n p_i a_i \cdot \sum_{i=1}^n p_i x_i \right\| \leq \frac{1}{4} |A - a| \|X - x\|. \quad (5.13)$$

*The constant  $\frac{1}{4}$  is sharp.*

**Proof.** A simple computation shows that

$$\sum_{i=1}^n p_i a_i x_i - \sum_{i=1}^n p_i a_i \sum_{i=1}^n p_i x_i = \frac{1}{2} \sum_{i,j=1}^n p_i p_j (a_i - a_j) (x_i - x_j). \quad (5.14)$$

Taking the norm in both parts of (5.14) and using the generalized triangle inequality, we obtain

$$\left\| \sum_{i=1}^n p_i a_i x_i - \sum_{i=1}^n p_i a_i \sum_{i=1}^n p_i x_i \right\| \leq \frac{1}{2} \sum_{i,j=1}^n p_i p_j |a_i - a_j| \|x_i - x_j\|. \quad (5.15)$$

By the Cauchy-Bunyakovsky-Schwartz discrete inequality for double sums, we obtain

$$\begin{aligned} & \left( \frac{1}{2} \sum_{i,j=1}^n p_i p_j |a_i - a_j| \|x_i - x_j\| \right)^2 \\ & \leq \left( \frac{1}{2} \sum_{i,j=1}^n p_i p_j |a_i - a_j|^2 \right) \left( \frac{1}{2} \sum_{i,j=1}^n p_i p_j \|x_i - x_j\|^2 \right). \end{aligned} \quad (5.16)$$

As a simple calculation reveals that

$$\frac{1}{2} \sum_{i,j=1}^n p_i p_j |a_i - a_j|^2 = \sum_{i=1}^n p_i |a_i|^2 - \left| \sum_{i=1}^n p_i a_i \right|^2$$

and

$$\frac{1}{2} \sum_{i,j=1}^n p_i p_j \|x_i - x_j\|^2 = \sum_{i=1}^n p_i \|x_i\|^2 - \left\| \sum_{i=1}^n p_i x_i \right\|^2,$$

then, by (5.15) and (5.16), we conclude that

$$\begin{aligned} & \left\| \sum_{i=1}^n p_i a_i x_i - \sum_{i=1}^n p_i a_i \sum_{i=1}^n p_i x_i \right\| \\ & \leq \left( \sum_{i=1}^n p_i |a_i|^2 - \left| \sum_{i=1}^n p_i a_i \right|^2 \right)^{\frac{1}{2}} \left( \sum_{i=1}^n p_i \|x_i\|^2 - \left\| \sum_{i=1}^n p_i x_i \right\|^2 \right)^{\frac{1}{2}}. \end{aligned} \quad (5.17)$$

However, from Lemma 199 and Corollary 201, we know that

$$\left( \sum_{i=1}^n p_i \|x_i\|^2 - \left\| \sum_{i=1}^n p_i x_i \right\|^2 \right)^{\frac{1}{2}} \leq \frac{1}{2} \|X - x\| \quad (5.18)$$

and

$$\left( \sum_{i=1}^n p_i |a_i|^2 - \left| \sum_{i=1}^n p_i a_i \right|^2 \right)^{\frac{1}{2}} \leq \frac{1}{2} |A - a|. \quad (5.19)$$

Consequently, by using (5.17) – (5.19), we deduce the desired estimate (5.13).

To prove the sharpness of the constant  $\frac{1}{4}$ , assume that (5.13) holds with a constant  $c > 0$ , i.e.,

$$\left\| \sum_{i=1}^n p_i a_i x_i - \sum_{i=1}^n p_i a_i \sum_{i=1}^n p_i x_i \right\| \leq c |A - a| \|X - x\| \quad (5.20)$$

for all  $p_i, a_i, x_i, a, A, x, X$  and  $n$  as in the hypothesis of Theorem 203.



If we choose  $n = 2$ ,  $a_1 = a$ ,  $a_2 = A$ ,  $x_1 = x$ ,  $x_2 = X$  ( $a \neq A$ ,  $x \neq X$ ) and  $p_1 = p_2 = \frac{1}{2}$ , then

$$\begin{aligned} \sum_{i=1}^2 p_i a_i x_i - \sum_{i=1}^2 p_i a_i \sum_{i=1}^2 p_i x_i &= \frac{1}{2} \sum_{i,j=1}^2 p_i p_j (a_i - a_j) (x_i - x_j) \\ &= \frac{1}{4} (a - A) (x - X). \end{aligned}$$

Consequently, from (5.20), we deduce

$$\frac{1}{4} |a - A| \|X - x\| \leq c |A - a| \|X - x\|,$$

which implies that  $c \geq \frac{1}{4}$ , and the theorem is completely proved. ■

**Remark 204** The condition (5.12) can be replaced by the more general assumption

$$\sum_{i=1}^n p_i \operatorname{Re} [(A - a_i) (\bar{a}_i - \bar{a})] \geq 0, \quad \sum_{i=1}^n p_i \operatorname{Re} \langle X - x_i, x_i - x \rangle \geq 0$$

and the conclusion (5.13) will still be valid.

The following corollary for real or complex numbers holds.

**Corollary 205** Let  $a_i, b_i \in \mathbb{K}$  ( $\mathbb{K} = \mathbb{C}, \mathbb{R}$ ),  $p_i \geq 0$  ( $i = 1, \dots, n$ ) with  $\sum_{i=1}^n p_i = 1$ . If  $a, A, b, B \in \mathbb{K}$  are such that

$$\operatorname{Re} [(A - a_i) (\bar{a}_i - \bar{a})] \geq 0, \quad \operatorname{Re} [(B - b_i) (\bar{b}_i - \bar{b})] \geq 0, \quad (5.21)$$

then we have the inequality

$$0 \leq \left| \sum_{i=1}^n p_i a_i b_i - \sum_{i=1}^n p_i a_i \sum_{i=1}^n p_i b_i \right| \leq \frac{1}{4} |A - a| |B - b|, \quad (5.22)$$

where the constant  $\frac{1}{4}$  is sharp.

**Remark 206** If we assume that  $a_i, b_i, a, A, b, B$  are real numbers, then (5.21) is equivalent to

$$a \leq a_i \leq A, b \leq b_i \leq B \text{ for all } i \in \{1, \dots, n\},$$

and (5.22) becomes

$$0 \leq \left| \sum_{i=1}^n p_i a_i b_i - \sum_{i=1}^n p_i a_i \sum_{i=1}^n p_i b_i \right| \leq \frac{1}{4} (A - a) (B - b),$$

which is the classical Grüss inequality for sequences of real numbers.

### 5.2.3 Applications for Discrete Fourier Transforms

Let  $(H; \langle \cdot, \cdot \rangle)$  be an inner product space over  $\mathbb{K}$  and  $\bar{x} = (x_1, \dots, x_n)$  be a sequence of vectors in  $H$ .

For a given  $w \in \mathbb{R}$ , define the *discrete Fourier transform* as

$$\mathcal{F}_w(\bar{x})(m) := \sum_{k=1}^n \exp(2wimk) \times x_k, \quad m = 1, \dots, n. \quad (5.23)$$

The following approximation result for the Fourier transform (5.23) holds [1].

**Theorem 207** *Let  $(H; \langle \cdot, \cdot \rangle)$  and  $\bar{x} \in H^n$  be as above. If there exists the vectors  $x, X \in H$  such that*

$$\operatorname{Re} \langle X - x_k, x_k - x \rangle \geq 0 \text{ for all } k \in \{1, \dots, n\}, \quad (5.24)$$

*then we have the inequality*

$$\begin{aligned} \left\| \mathcal{F}_w(\bar{x})(m) - \frac{\sin(wmn)}{\sin(wm)} \exp[w(n+1)im] \times \frac{1}{n} \sum_{k=1}^n x_k \right\| \\ \leq \frac{1}{2} \|X - x\| \left[ n^2 - \frac{\sin^2(wmn)}{\sin^2(wm)} \right]^{\frac{1}{2}}, \end{aligned} \quad (5.25)$$

for all  $m \in \{1, \dots, n\}$  and  $w \in \mathbb{R}$ ,  $w \neq \frac{l}{m}\pi$ ,  $l \in \mathbb{Z}$ .

**Proof.** From the inequality (5.17) in Theorem 203, we can state that

$$\begin{aligned} \left\| \frac{1}{n} \sum_{k=1}^n a_k x_k - \frac{1}{n} \sum_{k=1}^n a_k \cdot \frac{1}{n} \sum_{k=1}^n x_k \right\| \\ \leq \left( \frac{1}{n} \sum_{k=1}^n |a_k|^2 - \left| \frac{1}{n} \sum_{k=1}^n a_k \right|^2 \right)^{\frac{1}{2}} \left( \frac{1}{n} \sum_{k=1}^n \|x_k\|^2 - \left\| \frac{1}{n} \sum_{k=1}^n x_k \right\|^2 \right)^{\frac{1}{2}} \end{aligned} \quad (5.26)$$

for all  $a_k \in \mathbb{K}$ ,  $x_k \in H$  ( $k = 1, \dots, n$ ).

However, the  $x_k$  ( $k = 1, \dots, n$ ) satisfy (5.24), and therefore, by Lemma 199, we have

$$0 \leq \frac{1}{n} \sum_{k=1}^n \|x_k\|^2 - \left\| \frac{1}{n} \sum_{k=1}^n x_k \right\|^2 \leq \frac{1}{4} \|X - x\|^2. \quad (5.27)$$

Consequently, by (5.26) and (5.27), we conclude that

$$\begin{aligned} \left\| \sum_{k=1}^n a_k x_k - \sum_{k=1}^n a_k \cdot \frac{1}{n} \sum_{k=1}^n x_k \right\| \\ \leq \frac{1}{2} \|X - x\| \left( n \sum_{k=1}^n |a_k|^2 - \left| \sum_{k=1}^n a_k \right|^2 \right)^{\frac{1}{2}} \end{aligned} \quad (5.28)$$

for all  $a_k \in \mathbb{K}$  ( $k = 1, \dots, n$ ).

We now choose in (5.28),  $a_k = \exp(2wimk)$  to obtain

$$\begin{aligned} & \left\| \mathcal{F}_w(\bar{\mathbf{x}})(m) - \sum_{k=1}^n \exp(2wimk) \times \frac{1}{n} \sum_{k=1}^n x_k \right\| \\ & \leq \frac{1}{2} \|X - x\| \left( n \sum_{k=1}^n |\exp(2wimk)|^2 - \left| \sum_{k=1}^n \exp(2wimk) \right|^2 \right)^{\frac{1}{2}} \end{aligned} \quad (5.29)$$

for all  $m \in \{1, \dots, n\}$ .

As a simple calculation reveals that

$$\begin{aligned} \sum_{k=1}^n \exp(2wimk) &= \exp(2wim) \times \left[ \frac{\exp(2wimn) - 1}{\exp(2wim) - 1} \right] \\ &= \exp(2wim) \times \left[ \frac{\cos(2wmn) + i \sin(2wmn) - 1}{\cos(2wm) + i \sin(2wm) - 1} \right] \\ &= \exp(2wim) \times \frac{\sin(wmn)}{\sin(wm)} \left[ \frac{\cos(wmn) + i \sin(wmn)}{\cos(wm) + i \sin(wm)} \right] \\ &= \frac{\sin(wmn)}{\sin(wm)} \times \exp(2wim) \left[ \frac{\exp(iwmn)}{\exp(iwm)} \right] \\ &= \frac{\sin(wmn)}{\sin(wm)} \times \exp[w(n+1)im], \\ & \sum_{k=1}^n |\exp(2wimk)|^2 = n \end{aligned}$$

and

$$\left| \sum_{k=1}^n \exp(2wimk) \right|^2 = \frac{\sin^2(wmn)}{\sin^2(wm)}, \text{ for } w \neq \frac{l}{m}\pi, l \in \mathbb{Z},$$

thus, from (5.29), we deduce the desired inequality (5.25). ■

**Remark 208** *The assumption (5.24) can be replaced by the more general condition*

$$\sum_{i=1}^n \operatorname{Re} \langle X - x_i, x_i - x \rangle \geq 0,$$

*and the conclusion (5.25) will still remain valid.*

The following corollary is an obvious consequence of (5.25).

**Corollary 209** *Let  $a_i \in \mathbb{K}$  ( $i = 1, \dots, n$ ). If  $a, A \in \mathbb{K}$  are such that*

$$\operatorname{Re}[(A - a_i)(\bar{a}_i - \bar{a})] \geq 0 \text{ for all } i \in \{1, \dots, n\}, \quad (5.30)$$

then we have an approximation of the Fourier transform for the vector  $\bar{\mathbf{a}} = (a_1, \dots, a_n) \in \mathbb{K}^n$  :

$$\left\| \mathcal{F}_w(\bar{\mathbf{a}})(m) - \frac{\sin(wmn)}{\sin(wm)} \exp[w(n+1)im] \times \frac{1}{n} \sum_{k=1}^n a_k \right\| \leq \frac{1}{2} |A - a| \left[ n^2 - \frac{\sin^2(wmn)}{\sin^2(wm)} \right]^{\frac{1}{2}}, \quad (5.31)$$

for all  $m \in \{1, \dots, n\}$  and  $w \in \mathbb{R}$  so that  $w \neq \frac{l}{m}\pi$ ,  $l \in \mathbb{Z}$ .

**Remark 210** If we assume that  $\mathbb{K} = \mathbb{R}$ , then (5.30) is equivalent to

$$a \leq a_i \leq A \text{ for all } i \in \{1, \dots, n\}. \quad (5.32)$$

Consequently, with the assumption (5.32), we obtain the following approximation of the Fourier transform

$$\left\| \mathcal{F}_w(\bar{\mathbf{a}})(m) - \frac{\sin(wmn)}{\sin(wm)} \exp[w(n+1)im] \times \frac{1}{n} \sum_{k=1}^n a_k \right\| \leq \frac{1}{2} (A - a) \left[ n^2 - \frac{\sin^2(wmn)}{\sin^2(wm)} \right]^{\frac{1}{2}},$$

for all  $m \in \{1, \dots, n\}$  and  $w \neq \frac{l}{m}\pi$ ,  $l \in \mathbb{Z}$ .

#### 5.2.4 Applications for the Discrete Mellin Transform

Let  $(H; \langle \cdot, \cdot \rangle)$  be an inner product over  $\mathbb{R}$  and  $\bar{\mathbf{x}} = (x_1, \dots, x_n)$  be a sequence of vectors in  $H$ .

Define the *Mellin transform*:

$$\mathcal{M}(\bar{\mathbf{x}})(m) := \sum_{k=1}^n k^{m-1} x_k, \quad m = 1, \dots, n;$$

of the sequence  $\bar{\mathbf{x}} \in H^n$ .

The following approximation result holds [1].

**Theorem 211** Let  $H$  and  $\bar{\mathbf{x}}$  be as above. If there exist the vectors  $x, X \in H$  such that

$$\operatorname{Re} \langle X - x_k, x_k - x \rangle \geq 0 \text{ for all } k = 1, \dots, n; \quad (5.33)$$

then we have the inequality

$$\left\| \mathcal{M}(\bar{\mathbf{x}})(m) - S_{m-1}(n) \cdot \frac{1}{n} \sum_{k=1}^n x_k \right\| \leq \frac{1}{2} \|X - x\| \left[ n S_{2m-2}(n) - S_{m-1}^2(n) \right]^{\frac{1}{2}}, \quad m \in \{1, \dots, n\}, \quad (5.34)$$

where  $S_p(n)$ ,  $p \in \mathbb{R}$ ,  $n \in \mathbb{N}$  is the  $p$ -powered sum of the first  $n$  natural numbers, i.e.,

$$S_p(n) := \sum_{k=1}^n k^p.$$

**Proof.** We apply the inequality (5.28) to obtain

$$\begin{aligned} & \left\| \sum_{k=1}^n k^{m-1} x_k - \sum_{k=1}^n k^{m-1} \cdot \frac{1}{n} \sum_{k=1}^n x_k \right\| \\ & \leq \frac{1}{2} \|X - x\| \left[ n \sum_{k=1}^n k^{2(m-1)} - \left( \sum_{k=1}^n k^{m-1} \right)^2 \right]^{\frac{1}{2}} \\ & = \frac{1}{2} \|X - x\| [n S_{2m-2}(n) - S_{m-1}^2(n)]^{\frac{1}{2}}, \end{aligned}$$

and the inequality (5.34) is proved. ■

Consider the following particular values of Mellin Transform

$$\mu_1(\bar{\mathbf{x}}) := \sum_{k=1}^n k x_k$$

and

$$\mu_2(\bar{\mathbf{x}}) := \sum_{k=1}^n k^2 x_k.$$

The following corollary holds.

**Corollary 212** *Let  $H$  and  $\bar{\mathbf{x}}$  be as in Theorem 211. Then we have the inequalities:*

$$\left\| \mu_1(\bar{\mathbf{x}}) - \frac{n+1}{2} \cdot \sum_{k=1}^n x_k \right\| \leq \frac{1}{2} \|X - x\| n \left[ \frac{n(n+1)}{2} \right]^{\frac{1}{2}} \quad (5.35)$$

and

$$\begin{aligned} & \left\| \mu_2(\bar{\mathbf{x}}) - \frac{(n+1)(2n+1)}{6} \cdot \sum_{k=1}^n x_k \right\| \\ & \leq \frac{1}{12\sqrt{5}} \|X - x\| n \sqrt{(n-1)(n+1)(2n+1)(8n+1)}. \quad (5.36) \end{aligned}$$

**Remark 213** *If we assume that  $\bar{\mathbf{p}} = (p_1, \dots, p_n)$  is a probability distribution, i.e.,  $p_k \geq 0$  ( $k = 1, \dots, n$ ) and  $\sum_{k=1}^n p_k = 1$  and  $p \leq p_k \leq P$  ( $k = 1, \dots, n$ ), then by (5.35) and (5.36), we get the inequalities*

$$\left| \sum_{k=1}^n k p_k - \frac{n+1}{2} \right| \leq \frac{1}{2} (P - p) n \left[ \frac{n(n+1)}{2} \right]^{\frac{1}{2}}$$

and

$$\left| \sum_{k=1}^n k^2 p_k - \frac{(n+1)(2n+1)}{6} \right| \leq \frac{1}{12\sqrt{5}} (P-p) n \sqrt{(n-1)(n+1)(2n+1)(8n+1)}.$$

### 5.2.5 Applications for Polynomials

Let  $(H; \langle \cdot, \cdot \rangle)$  be an inner product space over  $\mathbb{K}$  and  $\bar{c} = (c_0, \dots, c_n)$  be a sequence of vectors in  $H$ .

Define the polynomial  $P: \mathbb{C} \rightarrow H$  with the coefficients  $\bar{c} = (c_0, \dots, c_n)$  by

$$P(z) = c_0 + zc_1 + \dots + z^n c_n, \quad z \in \mathbb{C}, \quad c_n \neq 0.$$

The following approximation result for the polynomial  $P$  holds [1].

**Theorem 214** *Let  $H, \bar{c}$  and  $P$  be as above. If there exist the vectors  $c, C \in H$  such that*

$$\operatorname{Re} \langle C - c_k, c_k - c \rangle \geq 0 \text{ for all } k \in \{0, \dots, n\}, \quad (5.37)$$

*then we have the inequality*

$$\begin{aligned} & \left\| P(z) - \frac{z^{n+1} - 1}{z - 1} \times \frac{c_0 + c_1 + \dots + c_n}{n+1} \right\| \\ & \leq \frac{1}{2} \|C - c\| \left[ (n+1) \frac{|z|^{2n+2} - 1}{|z|^2 - 1} - \frac{|z|^{2n+2} - 2 \operatorname{Re}(z^{n+1}) + 1}{|z|^2 - 2 \operatorname{Re}(z) + 1} \right]^{\frac{1}{2}} \end{aligned} \quad (5.38)$$

for all  $z \in \mathbb{C}$ ,  $|z| \neq 1$ .

**Proof.** Using the inequality (5.28), we can state that

$$\begin{aligned} & \left\| \sum_{k=0}^n z^k c_k - \sum_{k=0}^n z^k \cdot \frac{1}{n+1} \sum_{k=0}^n c_k \right\| \\ & \leq \frac{1}{2} \|C - c\| \left( (n+1) \sum_{k=0}^n |z|^{2k} - \left| \sum_{k=0}^n z^k \right|^2 \right)^{\frac{1}{2}} \\ & = \frac{1}{2} \|C - c\| \left[ (n+1) \frac{|z|^{2n+2} - 1}{|z|^2 - 1} - \left| \frac{z^{n+1} - 1}{z - 1} \right|^2 \right]^{\frac{1}{2}} \\ & = \frac{1}{2} \|C - c\| \left[ (n+1) \frac{|z|^{2n+2} - 1}{|z|^2 - 1} - \frac{|z|^{2n+2} - 2 \operatorname{Re}(z^{n+1}) + 1}{|z|^2 - 2 \operatorname{Re}(z) + 1} \right]^{\frac{1}{2}} \end{aligned} \quad (5.39)$$

and the inequality (5.38) is proved. ■

The following result for the complex roots of the unity also holds [1].

**Theorem 215** Let  $z_k := \cos\left(\frac{k\pi}{n+1}\right) + i \sin\left(\frac{k\pi}{n+1}\right)$ ,  $k \in \{0, \dots, n\}$  be the complex  $(n+1)$ -roots of the unity. Then we have the inequality

$$\|P(z_k)\| \leq \frac{1}{2}(n+1)\|C - c\|, \quad k \in \{1, \dots, n\}; \quad (5.40)$$

where the coefficients  $\bar{c} = (c_0, \dots, c_n) \in H^{n+1}$  satisfy the assumption (5.37).

**Proof.** From the inequality (5.39), we can state that

$$\begin{aligned} \left\| P(z_k) - \frac{z^{n+1} - 1}{z - 1} \cdot \frac{1}{n+1} \sum_{k=0}^n c_k \right\| \\ \leq \frac{1}{2} \|C - c\| \left[ (n+1) \sum_{k=0}^n |z|^{2k} - \left| \frac{z^{n+1} - 1}{z - 1} \right|^2 \right]^{\frac{1}{2}} \end{aligned}$$

for all  $z \in \mathbb{C}$ ,  $z \neq 1$ .

Putting  $z = z_k$ ,  $k \in \{1, \dots, n\}$  and taking into account that  $z_k^{n+1} = 1$ ,  $|z_k| = 1$ , we get the desired result (5.40). ■

The following corollary is a natural consequence of Theorem 215.

**Corollary 216** Let  $P(z) := \sum_{k=0}^n a_k z^k$  be a polynomial with real coefficients and  $z_k$  the  $(n+1)$ -roots of the unity as defined above. If  $a \leq a_k \leq A$ ,  $k = 0, \dots, n$ , then we have the inequality:

$$|P(z_k)| \leq \frac{1}{2}(n+1)(A - a).$$

### 5.2.6 Applications for Lipschitzian Mappings

Let  $(H; \langle \cdot, \cdot \rangle)$  be as above and  $F : H \rightarrow B$  a mapping defined on the inner product space  $H$  with values in the normed linear space  $B$  which satisfy the *Lipschitzian condition*:

$$|F(x) - F(y)| \leq L \|x - y\|, \quad \text{for all } x, y \in H, \quad (5.41)$$

where  $|\cdot|$  denotes the norm on  $B$  and  $\|\cdot\|$  is the Euclidean norm on  $H$ .

The following theorem holds [1].

**Theorem 217** Let  $F : H \rightarrow B$  be as above and  $x_i \in H$ ,  $p_i \geq 0$  ( $i = 1, \dots, n$ ) with  $P_n := \sum_{i=1}^n p_i > 0$ . If there exists two vectors  $x, X \in H$  such that

$$\operatorname{Re} \langle X - x_i, x_i - x \rangle \geq 0 \quad \text{for all } i \in \{1, \dots, n\}, \quad (5.42)$$

then we have the inequality

$$\left| \frac{1}{P_n} \sum_{i=1}^n p_i F(x_i) - F\left(\frac{1}{P_n} \sum_{i=1}^n p_i x_i\right) \right| \leq \frac{1}{2} \cdot L \|X - x\|. \quad (5.43)$$

**Proof.** As  $F$  is Lipschitzian, we have (5.41) for all  $x, y \in H$ . Choose  $x = \frac{1}{P_n} \sum_{i=1}^n p_i x_i$  and  $y = x_j$  ( $j = 1, \dots, n$ ), to get

$$\left| F\left(\frac{1}{P_n} \sum_{i=1}^n p_i x_i\right) - F(x_j) \right| \leq L \left\| \frac{1}{P_n} \sum_{i=1}^n p_i x_i - x_j \right\|, \quad (5.44)$$

for all  $j \in \{1, \dots, n\}$ .

If we multiply (5.44) by  $p_j \geq 0$  and sum over  $j$  from 1 to  $n$ , we obtain

$$\sum_{j=1}^n p_j \left| F\left(\frac{1}{P_n} \sum_{i=1}^n p_i x_i\right) - F(x_j) \right| \leq L \sum_{j=1}^n p_j \left\| \frac{1}{P_n} \sum_{i=1}^n p_i x_i - x_j \right\|. \quad (5.45)$$

Using the generalized triangle inequality, we have

$$\begin{aligned} \sum_{j=1}^n p_j \left| F\left(\frac{1}{P_n} \sum_{i=1}^n p_i x_i\right) - F(x_j) \right| \\ \geq \left| P_n F\left(\frac{1}{P_n} \sum_{i=1}^n p_i x_i\right) - \sum_{j=1}^n p_j F(x_j) \right|. \end{aligned} \quad (5.46)$$

By the Cauchy-Buniakowsky-Schwarz inequality, we also have

$$\begin{aligned} \sum_{j=1}^n p_j \left\| \frac{1}{P_n} \sum_{i=1}^n p_i x_i - x_j \right\| \\ \leq \left[ \sum_{j=1}^n p_j \left\| \frac{1}{P_n} \sum_{i=1}^n p_i x_i - x_j \right\|^2 \right]^{\frac{1}{2}} P_n^{\frac{1}{2}} \\ = P_n^{\frac{1}{2}} \left[ \sum_{j=1}^n p_j \left[ \left\| \frac{1}{P_n} \sum_{i=1}^n p_i x_i \right\|^2 - 2 \operatorname{Re} \left\langle \frac{1}{P_n} \sum_{i=1}^n p_i x_i, x_j \right\rangle + \|x_j\|^2 \right] \right]^{\frac{1}{2}} \\ = P_n^{\frac{1}{2}} \left[ P_n \left\| \frac{1}{P_n} \sum_{i=1}^n p_i x_i \right\|^2 - 2 \operatorname{Re} \left\langle \frac{1}{P_n} \sum_{i=1}^n p_i x_i, \sum_{j=1}^n p_j x_j \right\rangle + \sum_{j=1}^n p_j \|x_j\|^2 \right]^{\frac{1}{2}} \\ = P_n \left[ \frac{1}{P_n} \sum_{i=1}^n p_i \|x_i\|^2 - \left\| \frac{1}{P_n} \sum_{i=1}^n p_i x_i \right\|^2 \right]^{\frac{1}{2}}. \end{aligned} \quad (5.47)$$

Combining the above inequalities (5.45) – (5.47) we deduce, by dividing with



$P_n > 0$ , that

$$\begin{aligned} & \left| F \left( \frac{1}{P_n} \sum_{i=1}^n p_i x_i \right) - \frac{1}{P_n} \sum_{i=1}^n p_i F(x_i) \right| \\ & \leq L \cdot \left[ \frac{1}{P_n} \sum_{i=1}^n p_i \|x_i\|^2 - \left\| \frac{1}{P_n} \sum_{i=1}^n p_i x_i \right\|^2 \right]^{\frac{1}{2}}. \quad (5.48) \end{aligned}$$

Finally, using Lemma 199, we obtain the desired result. ■

**Remark 218** *The condition (5.42) can be substituted by the more general condition*

$$\sum_{i=1}^n p_i \operatorname{Re} \langle X - x_i, x_i - x \rangle \geq 0,$$

*and the conclusion (5.43) will still remain valid.*

The following corollary is a natural consequence of the above findings.

**Corollary 219** *Let  $x_i \in H$  ( $i = 1, \dots, n$ ) and  $x, X \in H$  be such that the condition (5.42) holds. Then we have the inequality*

$$0 \leq \frac{1}{P_n} \sum_{i=1}^n p_i \|x_i\| - \left\| \frac{1}{P_n} \sum_{i=1}^n p_i x_i \right\| \leq \frac{1}{2} \|X - x\|.$$

The proof follows by Theorem 217 by choosing  $F : H \rightarrow \mathbb{R}$ ,  $F(x) = \|x\|$  which is Lipschitzian with the constant  $L = 1$ , as

$$|F(x) - F(y)| = ||x| - |y|| \leq \|x - y\|,$$

for all  $x, y \in H$ .

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## 5.3 The Version for Inner-Products

### 5.3.1 A Discrete Inequality of Grüss Type

The following Grüss type inequality holds [1].

**Theorem 220** *Let  $(H; \langle \cdot, \cdot \rangle)$  be an inner product space over  $\mathbb{K}$ ;  $\mathbb{K} = \mathbb{C}, \mathbb{R}$ ,  $x_i, y_i \in H$ ,  $p_i \geq 0$  ( $i = 0, \dots, n$ ) ( $n \geq 2$ ) with  $\sum_{i=1}^n p_i = 1$ . If  $x, X, y, Y \in H$  are such that*

$$\operatorname{Re} \langle X - x_i, x_i - x \rangle \geq 0 \text{ and } \operatorname{Re} \langle Y - y_i, y_i - y \rangle \geq 0 \quad (5.49)$$

*for all  $i \in \{1, \dots, n\}$ , then we have the inequality*

$$\left| \sum_{i=1}^n p_i \langle x_i, y_i \rangle - \left\langle \sum_{i=1}^n p_i x_i, \sum_{i=1}^n p_i y_i \right\rangle \right| \leq \frac{1}{4} \|X - x\| \|Y - y\|. \quad (5.50)$$

*The constant  $\frac{1}{4}$  is sharp.*

**Proof.** A simple calculation shows that

$$\begin{aligned} \sum_{i=1}^n p_i \langle x_i, y_i \rangle - \left\langle \sum_{i=1}^n p_i x_i, \sum_{i=1}^n p_i y_i \right\rangle \\ = \frac{1}{2} \sum_{i,j=1}^n p_i p_j \langle x_i - x_j, y_i - y_j \rangle. \end{aligned} \quad (5.51)$$

Taking the modulus in both parts of (5.51) and using the generalized triangle inequality, we obtain

$$\left| \sum_{i=1}^n p_i \langle x_i, y_i \rangle - \left\langle \sum_{i=1}^n p_i x_i, \sum_{i=1}^n p_i y_i \right\rangle \right| \leq \frac{1}{2} \sum_{i,j=1}^n p_i p_j |\langle x_i - x_j, y_i - y_j \rangle|.$$

By Schwarz's inequality in inner product spaces we have

$$|\langle x_i - x_j, y_i - y_j \rangle| \leq \|x_i - x_j\| \|y_i - y_j\|,$$

for all  $i, j \in \{1, \dots, n\}$ , and therefore

$$\left| \sum_{i=1}^n p_i \langle x_i, y_i \rangle - \left\langle \sum_{i=1}^n p_i x_i, \sum_{i=1}^n p_i y_i \right\rangle \right| \leq \frac{1}{2} \sum_{i,j=1}^n p_i p_j \|x_i - x_j\| \|y_i - y_j\|.$$

Using the Cauchy-Bunyakovsky-Schwarz inequality for double sums, we can state that

$$\begin{aligned} \frac{1}{2} \sum_{i,j=1}^n p_i p_j \|x_i - x_j\| \|y_i - y_j\| \\ \leq \left( \frac{1}{2} \sum_{i,j=1}^n p_i p_j \|x_i - x_j\|^2 \right)^{\frac{1}{2}} \left( \frac{1}{2} \sum_{i,j=1}^n p_i p_j \|y_i - y_j\|^2 \right)^{\frac{1}{2}} \end{aligned}$$

and, as a simple calculation shows that,

$$\frac{1}{2} \sum_{i,j=1}^n p_i p_j \|x_i - x_j\|^2 = \sum_{i=1}^n p_i \|x_i\|^2 - \left\| \sum_{i=1}^n p_i x_i \right\|^2$$

and

$$\frac{1}{2} \sum_{i,j=1}^n p_i p_j \|y_i - y_j\|^2 = \sum_{i=1}^n p_i \|y_i\|^2 - \left\| \sum_{i=1}^n p_i y_i \right\|^2,$$

we obtain

$$\begin{aligned} & \left| \sum_{i=1}^n p_i \langle x_i, y_i \rangle - \left\langle \sum_{i=1}^n p_i x_i, \sum_{i=1}^n p_i y_i \right\rangle \right| \\ & \leq \left( \sum_{i=1}^n p_i \|x_i\|^2 - \left\| \sum_{i=1}^n p_i x_i \right\|^2 \right)^{\frac{1}{2}} \left( \sum_{i=1}^n p_i \|y_i\|^2 - \left\| \sum_{i=1}^n p_i y_i \right\|^2 \right)^{\frac{1}{2}}. \end{aligned} \quad (5.52)$$

Using Lemma 199, we know that

$$\left( \sum_{i=1}^n p_i \|x_i\|^2 - \left\| \sum_{i=1}^n p_i x_i \right\|^2 \right)^{\frac{1}{2}} \leq \frac{1}{2} \|X - x\|$$

and

$$\left( \sum_{i=1}^n p_i \|y_i\|^2 - \left\| \sum_{i=1}^n p_i y_i \right\|^2 \right)^{\frac{1}{2}} \leq \frac{1}{2} \|Y - y\|,$$

and then, by (5.52), we deduce the desired inequality (5.51).

To prove the sharpness of the constant  $\frac{1}{4}$ , let us assume that (5.50) holds with a constant  $c > 0$ , i.e.,

$$\left| \sum_{i=1}^n p_i \langle x_i, y_i \rangle - \left\langle \sum_{i=1}^n p_i x_i, \sum_{i=1}^n p_i y_i \right\rangle \right| \leq c \|X - x\| \|Y - y\| \quad (5.53)$$

under the above assumptions for  $p_i, x_i, y_i, x, X, y, Y$  and  $n \geq 2$ .

If we choose  $n = 2, x_1 = x, x_2 = X, y_1 = y, y_2 = Y$  ( $x \neq X, y \neq Y$ ) and  $p_1 = p_2 = \frac{1}{2}$ , then

$$\begin{aligned} \sum_{i=1}^2 p_i \langle x_i, y_i \rangle - \left\langle \sum_{i=1}^2 p_i x_i, \sum_{i=1}^2 p_i y_i \right\rangle &= \frac{1}{2} \sum_{i,j=1}^2 p_i p_j \langle x_i - x_j, y_i - y_j \rangle \\ &= \sum_{1 \leq i < j \leq 2} p_i p_j \langle x_i - x_j, y_i - y_j \rangle \\ &= \frac{1}{4} \langle x - X, y - Y \rangle \end{aligned}$$

and then

$$\left| \sum_{i=1}^2 p_i \langle x_i, y_i \rangle - \left\langle \sum_{i=1}^2 p_i x_i, \sum_{i=1}^2 p_i y_i \right\rangle \right| = \frac{1}{4} |\langle x - X, y - Y \rangle|.$$

Choose  $X - x = z$ ,  $Y - y = z$ ,  $z \neq 0$ . Then using (5.53), we derive

$$\frac{1}{4} \|z\|^2 \leq c \|z\|^2, \quad z \neq 0$$

which implies that  $c \geq \frac{1}{4}$ , and the theorem is proved. ■

**Remark 221** The condition (5.49) can be replaced by the more general assumption

$$\sum_{i=1}^n p_i \operatorname{Re} \langle X - x_i, x_i - x \rangle \geq 0, \quad \sum_{i=1}^n p_i \operatorname{Re} \langle Y - y_i, y_i - y \rangle \geq 0$$

and the conclusion (5.50) still remains valid.

The following corollary for real or complex numbers holds.

**Corollary 222** Let  $a_i, b_i \in \mathbb{K}$  ( $\mathbb{K} = \mathbb{C}, \mathbb{R}$ ),  $p_i \geq 0$  ( $i = 1, \dots, n$ ) with  $\sum_{i=1}^n p_i = 1$ . If  $a, A, b, B \in \mathbb{K}$  are such that

$$\operatorname{Re}[(A - a_i)(\bar{a}_i - \bar{a})] \geq 0, \quad \operatorname{Re}[(B - b_i)(\bar{b}_i - \bar{b})] \geq 0, \quad (5.54)$$

then we have the inequality

$$\left| \sum_{i=1}^n p_i a_i \bar{b}_i - \sum_{i=1}^n p_i a_i \sum_{i=1}^n p_i \bar{b}_i \right| \leq \frac{1}{4} |A - a| |B - b| \quad (5.55)$$

and the constant  $\frac{1}{4}$  is sharp.

The proof is obvious by Theorem 220 applied for the inner product space  $(\mathbb{C}, \langle \cdot, \cdot \rangle)$ , where  $\langle x, y \rangle = x \cdot \bar{y}$ . We omit the details.

**Remark 223** The condition (5.54) can be replaced by the more general condition

$$\sum_{i=1}^n p_i \operatorname{Re}[(A - a_i)(\bar{a}_i - \bar{a})] \geq 0, \quad \sum_{i=1}^n p_i \operatorname{Re}[(B - b_i)(\bar{b}_i - \bar{b})] \geq 0$$

and the conclusion of the above corollary will still remain valid.

**Remark 224** If we assume that  $a_i, b_i, a, b, A, B$  are real numbers, then (5.54) is equivalent to

$$a \leq a_i \leq A, \quad b \leq b_i \leq B \quad \text{for all } i \in \{1, \dots, n\}$$

and (5.55) becomes

$$0 \leq \left| \sum_{i=1}^n p_i a_i b_i - \sum_{i=1}^n p_i a_i \sum_{i=1}^n p_i b_i \right| \leq \frac{1}{4} (A - a)(B - b),$$

which is the classical Grüss inequality for sequences of real numbers.

### 5.3.2 Applications for Convex Functions

Let  $(H; \langle \cdot, \cdot \rangle)$  be a real inner product space and  $F : H \rightarrow \mathbb{R}$  a Fréchet differentiable convex mapping on  $H$ . Then we have the “*gradient inequality*”

$$F(x) - F(y) \geq \langle \nabla F(y), x - y \rangle \quad (5.56)$$

for all  $x, y \in H$ , where  $\nabla F : H \rightarrow H$  is the gradient operator associated to the differentiable convex function  $F$ .

The following theorem holds [1].

**Theorem 225** *Let  $F : H \rightarrow \mathbb{R}$  be as above and  $x_i \in H$  ( $i = 1, \dots, n$ ). Suppose that there exists the vectors  $x, X \in H$  such that  $\langle x_i - x, X - x_i \rangle \geq 0$  for all  $i \in \{1, \dots, m\}$  and  $y, Y \in H$  such that  $\langle \nabla F(x_i) - y, Y - \nabla F(x_i) \rangle \geq 0$  for all  $i \in \{1, \dots, m\}$ . Then for all  $p_i \geq 0$  ( $i = 1, \dots, m$ ) with  $P_m := \sum_{i=1}^m p_i > 0$ , we have the inequality*

$$0 \leq \frac{1}{P_m} \sum_{i=1}^m p_i F(x_i) - F\left(\frac{1}{P_m} \sum_{i=1}^m p_i x_i\right) \leq \frac{1}{4} \|X - x\| \|Y - y\|. \quad (5.57)$$

**Proof.** Choose in (5.56),  $x = \frac{1}{P_m} \sum_{i=1}^m p_i x_i$  and  $y = x_j$  to obtain

$$F\left(\frac{1}{P_m} \sum_{i=1}^m p_i x_i\right) - F(x_j) \geq \left\langle \nabla F(x_j), \frac{1}{P_m} \sum_{i=1}^m p_i x_i - x_j \right\rangle \quad (5.58)$$

for all  $j \in \{1, \dots, n\}$ .

If we multiply (5.58) by  $p_j \geq 0$  and sum over  $j$  from 1 to  $m$ , we have

$$\begin{aligned} P_m F\left(\frac{1}{P_m} \sum_{i=1}^m p_i x_i\right) - \sum_{j=1}^m p_j F(x_j) \\ \geq \frac{1}{P_m} \left\langle \sum_{j=1}^m \nabla F(x_j), \sum_{i=1}^m p_i x_i \right\rangle - \sum_{i=1}^m \langle \nabla F(x_j), x_j \rangle. \end{aligned}$$

Dividing by  $P_m > 0$ , we obtain the inequality

$$\begin{aligned} 0 &\leq \frac{1}{P_m} \sum_{i=1}^m p_i F(x_i) - F\left(\frac{1}{P_m} \sum_{i=1}^m p_i x_i\right) \\ &\leq \frac{1}{P_m} \sum_{i=1}^m p_i \langle \nabla F(x_i), x_i \rangle - \left\langle \frac{1}{P_m} \sum_{i=1}^m p_i \nabla F(x_i), \frac{1}{P_m} \sum_{i=1}^m p_i x_i \right\rangle, \end{aligned} \quad (5.59)$$

which is a generalisation for the case of inner product spaces of the result by Dragomir-Goh established in 1996 for the case of differentiable mappings defined on  $\mathbb{R}^n$  [2].

Applying Theorem 220 for real inner product spaces, and  $y_i = \nabla F(x_i)$ , we easily deduce

$$\begin{aligned} \frac{1}{P_m} \sum_{i=1}^m p_i \langle x_i, \nabla F(x_i) \rangle - \left\langle \frac{1}{P_m} \sum_{i=1}^m p_i x_i, \frac{1}{P_m} \sum_{i=1}^m p_i \nabla F(x_i) \right\rangle \\ \leq \frac{1}{4} \|X - x\| \|Y - y\| \quad (5.60) \end{aligned}$$

and then, by (5.59) and (5.60) we can conclude that the desired inequality (5.57) holds. ■

### 5.3.3 Applications for Some Discrete Transforms

Let  $(H; \langle \cdot, \cdot \rangle)$  be an inner product space over  $\mathbb{K}$ ,  $\mathbb{K} = \mathbb{C}, \mathbb{R}$  and  $\bar{\mathbf{x}} = (x_1, \dots, x_n)$  be a sequence of vectors in  $H$ .

For a given  $m \in \mathbb{K}$ , define the *discrete Fourier Transform*

$$\mathcal{F}_w(\bar{\mathbf{x}})(m) = \sum_{k=1}^n \exp(2wimk) \times x_k, \quad m = 1, \dots, n.$$

The complex number  $\sum_{k=1}^n \exp(2wimk) \langle x_k, y_k \rangle$  is actually the usual Fourier transform of the vector  $(\langle x_1, y_1 \rangle, \dots, \langle x_n, y_n \rangle) \in \mathbb{K}^n$  and will be denoted by

$$\mathcal{F}_w(\bar{\mathbf{x}} \cdot \bar{\mathbf{y}})(m) = \sum_{k=1}^n \exp(2wimk) \langle x_k, y_k \rangle, \quad m = 1, \dots, n.$$

The following result holds [1].

**Theorem 226** *Let  $\bar{\mathbf{x}}, \bar{\mathbf{y}} \in H^n$  be sequences of vectors such that there exists the vectors  $c, C, y, Y \in H$  with the properties*

$$\operatorname{Re} \langle C - \exp(2wimk) x_k, \exp(2wimk) x_k - c \rangle \geq 0, \quad k, m = 1, \dots, n$$

and

$$\operatorname{Re} \langle Y - y_k, y_k - y \rangle \geq 0, \quad k = 1, \dots, n. \quad (5.61)$$

Then we have the inequality

$$\left| \mathcal{F}_w(\bar{\mathbf{x}} \cdot \bar{\mathbf{y}})(m) - \left\langle \mathcal{F}_w(\bar{\mathbf{x}})(m), \frac{1}{n} \sum_{k=1}^n y_k \right\rangle \right| \leq \frac{n}{4} \|C - c\| \|Y - y\|,$$

for all  $m \in \{1, \dots, n\}$ .

The proof follows by Theorem 220 applied for  $p_k = \frac{1}{n}$  and for the sequences  $x_k \rightarrow c_k = \exp(2wimk) x_k$  and  $y_k$  ( $k = 1, \dots, n$ ). We omit the details.

We can also consider the *Mellin transform*

$$\mathcal{M}(\bar{\mathbf{x}})(m) := \sum_{k=1}^n k^{m-1} x_k, \quad m = 1, \dots, n,$$

of the sequence  $\bar{\mathbf{x}} = (x_1, \dots, x_n) \in H^n$ .

We remark that the complex number  $\sum_{k=1}^n k^{m-1} \langle x_k, y_k \rangle$  is actually the Mellin transform of the vector  $(\langle x_1, y_1 \rangle, \dots, \langle x_n, y_n \rangle) \in \mathbb{K}^n$  and will be denoted by

$$\mathcal{M}(\bar{\mathbf{x}} \cdot \bar{\mathbf{y}})(m) := \sum_{k=1}^n k^{m-1} \langle x_k, y_k \rangle.$$

The following theorem holds [1].

**Theorem 227** *Let  $\bar{\mathbf{x}}, \bar{\mathbf{y}} \in H^n$  be sequences of vectors such that there exist the vectors  $d, D, y, Y \in H$  with the properties*

$$\operatorname{Re} \langle D - k^{m-1} x_k, k^{m-1} x_k - d \rangle \geq 0$$

*for all  $k, m \in \{1, \dots, n\}$ , and (5.61) is fulfilled.*

*Then we have the inequality*

$$\left| \mathcal{M}(\bar{\mathbf{x}} \cdot \bar{\mathbf{y}})(m) - \left\langle \mathcal{M}(\bar{\mathbf{x}})(m), \frac{1}{n} \sum_{k=1}^n y_k \right\rangle \right| \leq \frac{n}{4} \|D - d\| \|Y - y\|$$

*for all  $m \in \{1, \dots, n\}$ .*

The proof follows by Theorem 220 applied for  $p_k = \frac{1}{n}$  and for the sequences  $x_k \rightarrow d_k = kx_k$  and  $y_k$  ( $k = 1, \dots, n$ ). We omit the details.

Another result which connects the Fourier transforms for different parameters  $w$  also holds [1].

**Theorem 228** *Let  $\bar{\mathbf{x}}, \bar{\mathbf{y}} \in H^n$  and  $w, z \in \mathbb{K}$ . If there exist the vectors  $e, E, f, F \in H$  such that*

$$\operatorname{Re} \langle E - \exp(2wimk) x_k, \exp(2wimk) x_k - e \rangle \geq 0, \quad k, m = 1, \dots, n$$

*and*

$$\operatorname{Re} \langle F - \exp(2zimbk) y_k, \exp(2zimbk) y_k - f \rangle \geq 0, \quad k, m = 1, \dots, n$$

*then we have the inequality:*

$$\left| \frac{1}{n} \mathcal{F}_{w+z}(\bar{\mathbf{x}} \cdot \bar{\mathbf{y}})(m) - \left\langle \frac{1}{n} \mathcal{F}_w(\bar{\mathbf{x}})(m), \frac{1}{n} \mathcal{F}_z(\bar{\mathbf{y}})(m) \right\rangle \right| \leq \frac{1}{4} \|E - e\| \|F - f\|,$$

*for all  $m \in \{1, \dots, n\}$ .*

The proof follows by Theorem 220 for the sequences  $\exp(2wimk) x_k, \exp(2zimbk) y_k$  ( $k = 1, \dots, n$ ). We omit the details.



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## 5.4 More Grüss' Type Inequalities

### 5.4.1 Introduction

In the recent paper [3], the author has obtained the following Grüss type inequality for forward difference.

**Theorem 229** *Let  $\bar{\mathbf{x}} = (x_1, \dots, x_n)$ ,  $\bar{\mathbf{y}} = (y_1, \dots, y_n) \in H^n$  and  $\bar{\mathbf{p}} \in \mathbb{R}_+^n$  be a probability sequence. Then one has the inequalities*

$$\left| \sum_{i=1}^n p_i \langle x_i, y_i \rangle - \left\langle \sum_{i=1}^n p_i x_i, \sum_{i=1}^n p_i y_i \right\rangle \right| \leq \begin{cases} \left[ \sum_{i=1}^n i^2 p_i - \left( \sum_{i=1}^n i p_i \right)^2 \right] \max_{k=1, \dots, n-1} \|\Delta x_k\| \max_{k=1, \dots, n-1} \|\Delta y_k\| ; \\ \sum_{1 \leq j < i \leq n} p_i p_j (i-j) \left( \sum_{k=1}^{n-1} \|\Delta x_k\|^p \right)^{\frac{1}{p}} \left( \sum_{k=1}^{n-1} \|\Delta y_k\|^q \right)^{\frac{1}{q}} \\ \text{if } p > 1, \frac{1}{p} + \frac{1}{q} = 1; \\ \frac{1}{2} \left[ \sum_{i=1}^n p_i (1-p_i) \right] \sum_{k=1}^{n-1} \|\Delta x_k\| \sum_{k=1}^{n-1} \|\Delta y_k\|. \end{cases} \quad (5.62)$$

The constants 1, 1 and  $\frac{1}{2}$  in the right hand side of the inequality (5.62) are best in the sense that they cannot be replaced by smaller constants.

If one chooses  $p_i = \frac{1}{n}$  ( $i = 1, \dots, n$ ) in (5.62), then the following unweighted inequalities hold:

$$\left| \frac{1}{n} \sum_{i=1}^n \langle x_i, y_i \rangle - \left\langle \frac{1}{n} \sum_{i=1}^n x_i, \frac{1}{n} \sum_{i=1}^n y_i \right\rangle \right| \leq \begin{cases} \frac{n^2-1}{12} \max_{k=1, \dots, n-1} \|\Delta x_k\| \max_{k=1, \dots, n-1} \|\Delta y_k\| ; \\ \frac{n^2-1}{6n} \left( \sum_{k=1}^{n-1} \|\Delta x_k\|^p \right)^{\frac{1}{p}} \left( \sum_{k=1}^{n-1} \|\Delta y_k\|^q \right)^{\frac{1}{q}} \\ \text{if } p > 1, \frac{1}{p} + \frac{1}{q} = 1; \\ \frac{n-1}{2n} \sum_{k=1}^{n-1} \|\Delta x_k\| \sum_{k=1}^{n-1} \|\Delta y_k\|. \end{cases} \quad (5.63)$$

Here, the constants  $\frac{1}{12}$ ,  $\frac{1}{6}$  and  $\frac{1}{2}$  are also best possible in the above sense.

The following reverse of the Cauchy-Bunyakovsky-Schwarz inequality for sequences of vectors in inner product spaces holds.

**Corollary 230** *With the assumptions in Theorem 229 for  $\overline{\mathbf{x}}$  and  $\overline{\mathbf{p}}$  one has the inequalities*

$$\begin{aligned}
0 &\leq \sum_{i=1}^n p_i \|x_i\|^2 - \left\| \sum_{i=1}^n p_i x_i \right\|^2 \\
&\leq \begin{cases} \left[ \sum_{i=1}^n i^2 p_i - \left( \sum_{i=1}^n i p_i \right)^2 \right] \max_{k=1, n-1} \|\Delta x_k\|^2; \\ \\ \sum_{1 \leq j < i \leq n} p_i p_j (i-j) \left( \sum_{k=1}^{n-1} \|\Delta x_k\|^p \right)^{\frac{1}{p}} \left( \sum_{k=1}^{n-1} \|\Delta x_k\|^q \right)^{\frac{1}{q}} \\ \quad \text{if } p > 1, \frac{1}{p} + \frac{1}{q} = 1; \\ \\ \frac{1}{2} \left[ \sum_{i=1}^n p_i (1-p_i) \right] \left( \sum_{k=1}^{n-1} \|\Delta x_k\| \right)^2. \end{cases}
\end{aligned} \tag{5.64}$$

The constants 1, 1 and  $\frac{1}{2}$  are best possible in the above sense.

The following particular inequalities that may be deduced from (5.64) on choosing the equal weights  $p_i = \frac{1}{n}$ ,  $i = 1, \dots, n$  are also of interest

$$\begin{aligned}
0 &\leq \frac{1}{n} \sum_{i=1}^n \|x_i\|^2 - \left\| \frac{1}{n} \sum_{i=1}^n x_i \right\|^2 \\
&\leq \begin{cases} \frac{n^2-1}{12} \max_{k=1, n-1} \|\Delta x_k\|^2; \\ \\ \frac{n^2-1}{6n} \left( \sum_{k=1}^{n-1} \|\Delta x_k\|^p \right)^{\frac{1}{p}} \left( \sum_{k=1}^{n-1} \|\Delta x_k\|^q \right)^{\frac{1}{q}} \\ \quad \text{if } p > 1, \frac{1}{p} + \frac{1}{q} = 1; \\ \\ \frac{n-1}{2n} \left( \sum_{k=1}^{n-1} \|\Delta x_k\| \right)^2. \end{cases}
\end{aligned} \tag{5.65}$$

Here the constants  $\frac{1}{12}$ ,  $\frac{1}{6}$  and  $\frac{1}{2}$  are also best possible.

The main aim of this section is to present, by following [5], a different class of Grüss type inequalities for sequences of vectors in inner product spaces and to apply them for obtaining a reverse of Jensen's inequality for convex functions defined on such spaces.

### 5.4.2 More Grüss Type Inequalities

The following lemma holds (see also [4]).

**Lemma 231** Let  $a, x, A$  be vectors in the inner product space  $(H; \langle \cdot, \cdot \rangle)$  over the real or complex number field  $\mathbb{K}$  ( $\mathbb{K} = \mathbb{R}, \mathbb{C}$ ) with  $a \neq A$ . The following statements are equivalent:

- (i)  $\operatorname{Re} \langle A - x, x - a \rangle \geq 0$ ;
- (ii)  $\left\| x - \frac{a+A}{2} \right\| \leq \frac{1}{2} \|A - a\|$ .

The following inequality of Grüss type for sequences of vectors in inner product spaces holds [5].

**Theorem 232** Let  $(H; \langle \cdot, \cdot \rangle)$  be an inner product over  $\mathbb{K}$  ( $\mathbb{K} = \mathbb{C}, \mathbb{R}$ ), and  $\bar{\mathbf{x}} = (x_1, \dots, x_n)$ ,  $\bar{\mathbf{y}} = (y_1, \dots, y_n) \in H^n$ ,  $\bar{\mathbf{p}} \in \mathbb{R}_+^n$  with  $\sum_{i=1}^n p_i = 1$ . If  $x, X \in H$  are such that

$$\operatorname{Re} \langle X - x_i, x_i - x \rangle \geq 0 \quad \text{for each } i \in \{1, \dots, n\}, \quad (5.66)$$

or equivalently,

$$\left\| x_i - \frac{x + X}{2} \right\| \leq \frac{1}{2} \|X - x\| \quad \text{for each } i \in \{1, \dots, n\}, \quad (5.67)$$

then one has the inequality

$$\begin{aligned} & \left| \sum_{i=1}^n p_i \langle x_i, y_i \rangle - \left\langle \sum_{i=1}^n p_i x_i, \sum_{i=1}^n p_i y_i \right\rangle \right| \\ & \leq \frac{1}{2} \|X - x\| \sum_{i=1}^n p_i \left\| y_i - \sum_{j=1}^n p_j y_j \right\| \\ & \leq \frac{1}{2} \|X - x\| \left[ \sum_{i=1}^n p_i \|y_i\|^2 - \left\| \sum_{i=1}^n p_i y_i \right\|^2 \right]^{\frac{1}{2}}. \end{aligned} \quad (5.68)$$

The constant  $\frac{1}{2}$  is best possible in the first and second inequality in the sense that it cannot be replaced by a smaller constant.

**Proof.** It is easy to see that the following identity holds true

$$\begin{aligned} & \sum_{i=1}^n p_i \langle x_i, y_i \rangle - \left\langle \sum_{i=1}^n p_i x_i, \sum_{i=1}^n p_i y_i \right\rangle \\ & = \sum_{i=1}^n p_i \left\langle x_i - \frac{x + X}{2}, y_i - \sum_{j=1}^n p_j y_j \right\rangle. \end{aligned} \quad (5.69)$$

Taking the modulus in (5.69) and using the Schwarz inequality in the inner product space  $(H; \langle \cdot, \cdot \rangle)$ , we have

$$\begin{aligned}
& \left| \sum_{i=1}^n p_i \langle x_i, y_i \rangle - \left\langle \sum_{i=1}^n p_i x_i, \sum_{i=1}^n p_i y_i \right\rangle \right| \\
& \leq \sum_{i=1}^n p_i \left| \left\langle x_i - \frac{x+X}{2}, y_i - \sum_{j=1}^n p_j y_j \right\rangle \right| \\
& \leq \sum_{i=1}^n p_i \left\| x_i - \frac{x+X}{2} \right\| \left\| y_i - \sum_{j=1}^n p_j y_j \right\| \\
& \leq \frac{1}{2} \|X - x\| \sum_{i=1}^n p_i \left\| y_i - \sum_{j=1}^n p_j y_j \right\|,
\end{aligned}$$

and the first inequality in (5.68) is proved.

Using the Cauchy-Bunyakovsky-Schwarz inequality for positive sequences and the calculation rules in inner product spaces, we have

$$\sum_{i=1}^n p_i \left\| y_i - \sum_{j=1}^n p_j y_j \right\| \leq \left[ \sum_{i=1}^n p_i \left\| y_i - \sum_{j=1}^n p_j y_j \right\|^2 \right]^{\frac{1}{2}}$$

and

$$\sum_{i=1}^n p_i \left\| y_i - \sum_{j=1}^n p_j y_j \right\|^2 = \sum_{i=1}^n p_i \|y_i\|^2 - \left\| \sum_{i=1}^n p_i y_i \right\|^2$$

giving the second part of (5.68).

To prove the sharpness of the constant  $\frac{1}{2}$  in the first inequality in (5.68), let us assume that, under the assumptions of the theorem, the inequality holds with a constant  $C > 0$ , i.e.,

$$\begin{aligned}
& \left| \sum_{i=1}^n p_i \langle x_i, y_i \rangle - \left\langle \sum_{i=1}^n p_i x_i, \sum_{i=1}^n p_i y_i \right\rangle \right| \\
& \leq C \|X - x\| \sum_{i=1}^n p_i \left\| y_i - \sum_{j=1}^n p_j y_j \right\|. \quad (5.70)
\end{aligned}$$

Consider  $n = 2$  and observe that

$$\begin{aligned}
& \sum_{i=1}^2 p_i \langle x_i, y_i \rangle - \left\langle \sum_{i=1}^2 p_i x_i, \sum_{i=1}^2 p_i y_i \right\rangle = p_2 p_1 \langle x_2 - x_1, y_2 - y_1 \rangle, \\
& \sum_{i=1}^2 p_i \left\| y_i - \sum_{j=1}^2 p_j y_j \right\| = 2 p_2 p_1 \|y_2 - y_1\|
\end{aligned}$$

and then, by (5.70), we deduce

$$p_2 p_1 |\langle x_2 - x_1, y_2 - y_1 \rangle| \leq 2C \|X - x\| p_2 p_1 \|y_2 - y_1\|. \quad (5.71)$$

If we choose  $p_1, p_2 > 0$ ,  $y_2 = x_2$ ,  $y_1 = x_1$  and  $x_2 = X$ ,  $x_1 = x$  with  $x \neq X$ , then (5.67) holds and from (5.71) we deduce  $C \geq \frac{1}{2}$ .

The fact that  $\frac{1}{2}$  is best possible in the second inequality may be proven in a similar manner and we omit the details. ■

**Remark 233** *If  $\bar{x}$  and  $\bar{y}$  satisfy the assumptions of Theorem 232, or equivalently*

$$\left\| x_i - \frac{x + X}{2} \right\| \leq \frac{1}{2} \|X - x\|, \quad \left\| y_i - \frac{y + Y}{2} \right\| \leq \frac{1}{2} \|Y - y\|, \quad (5.72)$$

for each  $i \in \{1, \dots, n\}$ , then by Theorem 232 we may state the following sequence of inequalities improving the Grüss inequality (5.69)

$$\begin{aligned} 0 &\leq \left| \sum_{i=1}^n p_i \langle x_i, y_i \rangle - \left\langle \sum_{i=1}^n p_i x_i, \sum_{i=1}^n p_i y_i \right\rangle \right| \\ &\leq \frac{1}{2} \|X - x\| \sum_{i=1}^n p_i \left\| y_i - \sum_{j=1}^n p_j y_j \right\| \\ &\leq \frac{1}{2} \|X - x\| \left( \sum_{i=1}^n p_i \|y_i\|^2 - \left\| \sum_{i=1}^n p_i y_i \right\|^2 \right)^{\frac{1}{2}} \\ &\leq \frac{1}{4} \|X - x\| \|Y - y\|. \end{aligned} \quad (5.73)$$

In particular, for  $x_i = y_i$  ( $i = 1, \dots, n$ ), one has

$$0 \leq \sum_{i=1}^n p_i \|x_i\|^2 - \left\| \sum_{i=1}^n p_i x_i \right\|^2 \leq \frac{1}{2} \|X - x\| \sum_{i=1}^n p_i \left\| x_i - \sum_{j=1}^n p_j x_j \right\| \quad (5.74)$$

and the constant  $\frac{1}{2}$  is best possible.

The following result also holds [5].

**Theorem 234** *Let  $(H; \langle \cdot, \cdot \rangle)$  and  $\mathbb{K}$  be as above and  $\bar{x} = (x_1, \dots, x_n) \in H^n$ ,  $\bar{\alpha} = (\alpha_1, \dots, \alpha_n) \in \mathbb{K}^n$  and  $\bar{p}$  a probability vector. If  $x, X \in H$  are such that*

(5.66) or equivalently, (5.67) holds, then we have the inequality

$$\begin{aligned}
0 &\leq \left\| \sum_{i=1}^n p_i \alpha_i x_i - \sum_{i=1}^n p_i \alpha_i \cdot \sum_{i=1}^n p_i x_i \right\| \\
&\leq \frac{1}{2} \|X - x\| \sum_{i=1}^n p_i \left| \alpha_i - \sum_{j=1}^n p_j \alpha_j \right| \\
&\leq \frac{1}{2} \|X - x\| \left[ \sum_{i=1}^n p_i |\alpha_i|^2 - \left| \sum_{i=1}^n p_i \alpha_i \right|^2 \right]^{\frac{1}{2}}.
\end{aligned} \tag{5.75}$$

The constant  $\frac{1}{2}$  in the first and second inequalities is best possible in the sense that it cannot be replaced by a smaller constant.

**Proof.** We start with the following equality that may be easily verified by direct calculation

$$\begin{aligned}
\sum_{i=1}^n p_i \alpha_i x_i - \sum_{i=1}^n p_i \alpha_i \cdot \sum_{i=1}^n p_i x_i \\
= \sum_{i=1}^n p_i \left( \alpha_i - \sum_{j=1}^n p_j \alpha_j \right) \left( x_i - \frac{x + X}{2} \right). \tag{5.76}
\end{aligned}$$

If we take the norm in (5.76), we deduce

$$\begin{aligned}
\left\| \sum_{i=1}^n p_i \alpha_i x_i - \sum_{i=1}^n p_i \alpha_i \cdot \sum_{i=1}^n p_i x_i \right\| &\leq \sum_{i=1}^n p_i \left| \alpha_i - \sum_{j=1}^n p_j \alpha_j \right| \left\| x_i - \frac{x + X}{2} \right\| \\
&\leq \frac{1}{2} \|X - x\| \sum_{i=1}^n p_i \left| \alpha_i - \sum_{j=1}^n p_j \alpha_j \right| \\
&\leq \frac{1}{2} \|X - x\| \left( \sum_{i=1}^n p_i \left( \alpha_i - \sum_{j=1}^n p_j \alpha_j \right)^2 \right)^{\frac{1}{2}} \\
&= \frac{1}{2} \|X - x\| \left( \sum_{i=1}^n p_i |\alpha_i|^2 - \left| \sum_{i=1}^n p_i \alpha_i \right|^2 \right)^{\frac{1}{2}},
\end{aligned}$$

proving the inequality (5.75).

The fact that the constant  $\frac{1}{2}$  is sharp may be proven in a similar manner to the one embodied in the proof of Theorem 232. We omit the details. ■

**Remark 235** If  $\bar{x}$  and  $\bar{\alpha}$  satisfy the assumption

$$\left\| \alpha_i - \frac{a+A}{2} \right\| \leq \frac{1}{2} |A-a|, \quad \left\| x_i - \frac{x+X}{2} \right\| \leq \frac{1}{2} \|X-x\|,$$

for each  $i \in \{1, \dots, n\}$ , then by Theorem 232 we may state the following sequence of inequalities improving the Grüss inequality

$$\begin{aligned} 0 &\leq \left\| \sum_{i=1}^n p_i \alpha_i x_i - \sum_{i=1}^n p_i \alpha_i \cdot \sum_{i=1}^n p_i x_i \right\| \\ &\leq \frac{1}{2} \|X-x\| \sum_{i=1}^n p_i \left| \alpha_i - \sum_{j=1}^n p_j \alpha_j \right| \\ &\leq \frac{1}{2} \|X-x\| \left( \sum_{i=1}^n p_i |\alpha_i|^2 - \left| \sum_{i=1}^n p_i \alpha_i \right|^2 \right)^{\frac{1}{2}} \\ &\leq \frac{1}{4} |A-a| \|X-x\|. \end{aligned}$$

**Remark 236** If in (5.75) we choose  $x_i = \alpha_i \in \mathbb{C}$  and assume that  $|\alpha_i - \frac{a+A}{2}| \leq \frac{1}{2} |A-a|$ , where  $a, A \in \mathbb{C}$ , then we get the following interesting inequality for complex numbers

$$\begin{aligned} 0 &\leq \left| \sum_{i=1}^n p_i \alpha_i^2 - \left( \sum_{i=1}^n p_i \alpha_i \right)^2 \right| \\ &\leq \frac{1}{2} |A-a| \sum_{i=1}^n p_i \left| \alpha_i - \sum_{j=1}^n p_j \alpha_j \right| \\ &\leq \frac{1}{2} |A-a| \left[ \sum_{i=1}^n p_i |\alpha_i|^2 - \left| \sum_{i=1}^n p_i \alpha_i \right|^2 \right]^{\frac{1}{2}}. \end{aligned}$$

### 5.4.3 Applications for Convex Functions

Let  $(H; \langle \cdot, \cdot \rangle)$  be a real inner product space and  $F : H \rightarrow \mathbb{R}$  a Fréchet differentiable convex function on  $H$ . If  $\nabla F : H \rightarrow H$  denotes the gradient operator associated to  $F$ , then we have the inequality

$$F(x) - F(y) \geq \langle \nabla F(y), x - y \rangle \quad (5.77)$$

for each  $x, y \in H$ .

The following result holds [5].



**Theorem 237** Let  $F : H \rightarrow \mathbb{R}$  be as above and  $z_i \in H, i \in \{1, \dots, n\}$ . Suppose that there exist the vectors  $m, M \in H$  such that either

$$\langle \nabla F(z_i) - m, M - \nabla F(z_i) \rangle \geq 0 \quad \text{for each } i \in \{1, \dots, n\};$$

or equivalently,

$$\left\| \nabla F(z_i) - \frac{m + M}{2} \right\| \leq \frac{1}{2} \|M - m\| \quad \text{for each } i \in \{1, \dots, n\}.$$

If  $q_i \geq 0$  ( $i \in \{1, \dots, n\}$ ) with  $Q_n := \sum_{i=1}^n q_i > 0$ , then we have the following converse of Jensen's inequality

$$\begin{aligned} 0 &\leq \frac{1}{Q_n} \sum_{i=1}^n q_i F(z_i) - F\left(\frac{1}{Q_n} \sum_{i=1}^n q_i z_i\right) \\ &\leq \frac{1}{2} \|M - m\| \frac{1}{Q_n} \sum_{i=1}^n q_i \left\| z_i - \frac{1}{Q_n} \sum_{j=1}^n q_j z_j \right\| \\ &\leq \frac{1}{2} \|M - m\| \left[ \frac{1}{Q_n} \sum_{i=1}^n q_i \|z_i\|^2 - \left\| \frac{1}{Q_n} \sum_{i=1}^n q_i z_i \right\|^2 \right]^{\frac{1}{2}}. \end{aligned} \quad (5.78)$$

**Proof.** We know, see for example [2, Eq. (4.4)], that the following reverse of Jensen's inequality for Fréchet differentiable convex functions

$$\begin{aligned} 0 &\leq \frac{1}{Q_n} \sum_{i=1}^n q_i F(z_i) - F\left(\frac{1}{Q_n} \sum_{i=1}^n q_i z_i\right) \\ &\leq \frac{1}{Q_n} \sum_{i=1}^n q_i \langle \nabla F(z_i), z_i \rangle - \left\langle \frac{1}{Q_n} \sum_{i=1}^n q_i \nabla F(z_i), \frac{1}{Q_n} \sum_{i=1}^n q_i z_i \right\rangle \end{aligned} \quad (5.79)$$

holds.

Now, if we use Theorem 232 for the choices  $x_i = \nabla F(z_i)$ ,  $y_i = z_i$  and  $p_i = \frac{1}{Q_n} q_i$ ,  $i \in \{1, \dots, n\}$ , we can state the inequality

$$\begin{aligned} &\frac{1}{Q_n} \sum_{i=1}^n q_i \langle \nabla F(z_i), z_i \rangle - \left\langle \frac{1}{Q_n} \sum_{i=1}^n q_i \nabla F(z_i), \frac{1}{Q_n} \sum_{i=1}^n q_i z_i \right\rangle \\ &\leq \frac{1}{2} \|M - m\| \frac{1}{Q_n} \sum_{i=1}^n q_i \left\| z_i - \frac{1}{Q_n} \sum_{j=1}^n q_j z_j \right\| \\ &\leq \frac{1}{2} \|M - m\| \left[ \frac{1}{Q_n} \sum_{i=1}^n q_i \|z_i\|^2 - \left\| \frac{1}{Q_n} \sum_{i=1}^n q_i z_i \right\|^2 \right]^{\frac{1}{2}}. \end{aligned} \quad (5.80)$$

Utilizing (5.79) and (5.80), we deduce the desired result (5.78). ■

If more information is available about the vector sequence  $\mathbf{z} = (z_1, \dots, z_n) \in H^n$ , then we may state the following corollary.

**Corollary 238** *With the assumptions in Theorem 237 and if there exist the vectors  $z, Z \in H$  such that either*

$$\langle z_i - z, Z - z_i \rangle \geq 0 \text{ for each } i \in \{1, \dots, n\}; \quad (5.81)$$

*or, equivalently*

$$\left\| z_i - \frac{z + Z}{2} \right\| \leq \frac{1}{2} \|Z - z\| \text{ for each } i \in \{1, \dots, n\}, \quad (5.82)$$

*then we have the inequality*

$$\begin{aligned} 0 &\leq \frac{1}{Q_n} \sum_{i=1}^n q_i F(z_i) - F\left(\frac{1}{Q_n} \sum_{i=1}^n q_i z_i\right) \\ &\leq \frac{1}{2} \|M - m\| \frac{1}{Q_n} \sum_{i=1}^n q_i \left\| z_i - \frac{1}{Q_n} \sum_{j=1}^n q_j z_j \right\| \\ &\leq \frac{1}{2} \|M - m\| \left[ \frac{1}{Q_n} \sum_{i=1}^n q_i \|z_i\|^2 - \left\| \frac{1}{Q_n} \sum_{i=1}^n q_i z_i \right\|^2 \right]^{\frac{1}{2}} \\ &\leq \frac{1}{4} \|M - m\| \|Z - z\|. \end{aligned} \quad (5.83)$$

**Remark 239** *Note that the inequality between the first term and the last term in (5.83) was first proved in [2, Theorem 4.1]. Consequently, the above corollary provides an improvement of the reverse of Jensen's inequality established in [2].*

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## 5.5 Some Inequalities for Forward Difference

### 5.5.1 Introduction

In [1], we have proved the following generalisation of the Grüss inequality.

**Theorem 240** *Let  $(H, \langle \cdot, \cdot \rangle)$  be an inner product space over  $\mathbb{K}$ ,  $\mathbb{K} = \mathbb{C}, \mathbb{R}$  and  $e \in H$ ,  $\|e\| = 1$ . If  $\phi, \Phi, \gamma, \Gamma \in \mathbb{K}$  and  $x, y \in H$  are such that*

$$\operatorname{Re} \langle \Phi e - x, x - \phi e \rangle \geq 0 \quad \text{and} \quad \operatorname{Re} \langle \Gamma e - y, y - \gamma e \rangle \geq 0$$

*hold, then we have the inequality*

$$|\langle x, y \rangle - \langle x, e \rangle \langle e, y \rangle| \leq \frac{1}{4} |\Phi - \phi| |\Gamma - \gamma|.$$

*The constant  $\frac{1}{4}$  is the best possible.*

A Grüss type inequality for sequences of vectors in inner product spaces was pointed out in [2].

**Theorem 241** *Let  $H$  and  $\mathbb{K}$  be as in Theorem 240 and  $x_i \in H$ ,  $a_i \in \mathbb{K}$ ,  $p_i \geq 0$  ( $i = 1, \dots, n$ ) ( $n \geq 2$ ) with  $\sum_{i=1}^n p_i = 1$ . If  $a, A \in \mathbb{K}$  and  $x, X \in H$  are such that:*

$$\operatorname{Re} [(A - a_i)(\overline{a_i} - \overline{a})] \geq 0, \quad \operatorname{Re} \langle X - x_i, x_i - x \rangle \geq 0$$

*for any  $i \in \{1, \dots, n\}$ , then we have the inequality*

$$\begin{aligned} 0 &\leq \left\| \sum_{i=1}^n p_i a_i x_i - \sum_{i=1}^n p_i a_i \cdot \sum_{i=1}^n p_i x_i \right\| \\ &\leq \frac{1}{4} |A - a| \|X - x\|. \end{aligned}$$

*The constant  $\frac{1}{4}$  is best possible.*

A complementary result for two sequences of vectors in inner product spaces is the following result that has been obtained in [3].

**Theorem 242** *Let  $H$  and  $\mathbb{K}$  be as above,  $x_i, y_i \in H$ ,  $p_i \geq 0$  ( $i = 1, \dots, n$ ) ( $n \geq 2$ ) with  $\sum_{i=1}^n p_i = 1$ . If  $x, X, y, Y \in H$  are such that:*

$$\operatorname{Re} \langle X - x_i, x_i - x \rangle \geq 0 \quad \text{and} \quad \operatorname{Re} \langle Y - y_i, y_i - y \rangle \geq 0$$

*for all  $i \in \{1, \dots, n\}$ , then we have the inequality*

$$0 \leq \left| \sum_{i=1}^n p_i \langle x_i, y_i \rangle - \left\langle \sum_{i=1}^n p_i x_i, \sum_{i=1}^n p_i y_i \right\rangle \right| \leq \frac{1}{4} \|X - x\| \|Y - y\|.$$

*The constant  $\frac{1}{4}$  is best possible.*

In the general case of normed linear spaces, the following Grüss type inequality in terms of the forward difference is known, see [4].

**Theorem 243** *Let  $(E, \|\cdot\|)$  be a normed linear space over  $\mathbb{K} = \mathbb{C}, \mathbb{R}$ ,  $x_i \in E$ ,  $\alpha_i \in \mathbb{K}$  and  $p_i \geq 0$  ( $i = 1, \dots, n$ ) such that  $\sum_{i=1}^n p_i = 1$ . Then we have the inequality*

$$\begin{aligned} 0 &\leq \left\| \sum_{i=1}^n p_i \alpha_i x_i - \sum_{i=1}^n p_i \alpha_i \cdot \sum_{i=1}^n p_i x_i \right\| \\ &\leq \max_{1 \leq j \leq n-1} |\Delta \alpha_j| \max_{1 \leq j \leq n-1} \|\Delta x_j\| \left[ \sum_{i=1}^n i^2 p_i - \left( \sum_{i=1}^n i p_i \right)^2 \right], \end{aligned} \quad (5.84)$$

where  $\Delta \alpha_j = \alpha_{j+1} - \alpha_j$  and  $\Delta x_j = x_{j+1} - x_j$  ( $j = 1, \dots, n-1$ ) are the forward differences of the vectors having the components  $\alpha_j$  and  $x_j$  ( $j = 1, \dots, n-1$ ), respectively.

The inequality (5.84) is sharp in the sense that the multiplicative constant  $C = 1$  in the right hand side cannot be replaced by a smaller one.

An important particular case is the one where all the weights are equal, giving the following corollary [4].

**Corollary 244** *Under the above assumptions for  $\alpha_i, x_i$  ( $i = 1, \dots, n$ ) we have the inequality*

$$\begin{aligned} 0 &\leq \left\| \frac{1}{n} \sum_{i=1}^n \alpha_i x_i - \frac{1}{n} \sum_{i=1}^n \alpha_i \cdot \frac{1}{n} \sum_{i=1}^n x_i \right\| \\ &\leq \frac{n^2 - 1}{12} \max_{1 \leq j \leq n-1} |\Delta \alpha_j| \max_{1 \leq j \leq n-1} \|\Delta x_j\|. \end{aligned} \quad (5.85)$$

The constant  $\frac{1}{12}$  is best possible.

Another result of this type was proved in [6].

**Theorem 245** *With the assumptions of Theorem 243, one has the inequality*

$$\begin{aligned} 0 &\leq \left\| \sum_{i=1}^n p_i \alpha_i x_i - \sum_{i=1}^n p_i \alpha_i \cdot \sum_{i=1}^n p_i x_i \right\| \\ &\leq \frac{1}{2} \sum_{j=1}^{n-1} |\Delta \alpha_j| \sum_{j=1}^{n-1} \|\Delta x_j\| \sum_{i=1}^n p_i (1 - p_i). \end{aligned} \quad (5.86)$$

The constant  $\frac{1}{2}$  is best possible.

As a useful particular case, we have the following corollary [6].

**Corollary 246** If  $\alpha_i, x_i$  ( $i = 1, \dots, n$ ) are as in Theorem 243, then

$$\begin{aligned} 0 &\leq \left\| \frac{1}{n} \sum_{i=1}^n \alpha_i x_i - \frac{1}{n} \sum_{i=1}^n \alpha_i \cdot \frac{1}{n} \sum_{i=1}^n x_i \right\| \\ &\leq \frac{1}{2} \left( 1 - \frac{1}{n} \right) \sum_{i=1}^{n-1} |\Delta \alpha_i| \sum_{i=1}^{n-1} \|\Delta x_i\|. \end{aligned}$$

The constant  $\frac{1}{2}$  is the best possible.

Finally, the following result is also known [5].

**Theorem 247** With the assumptions in Theorem 243, we have the inequality:

$$\begin{aligned} 0 &\leq \left\| \sum_{i=1}^n p_i \alpha_i x_i - \sum_{i=1}^n p_i \alpha_i \cdot \sum_{i=1}^n p_i x_i \right\| \\ &\leq \left( \sum_{j=1}^{n-1} |\Delta \alpha_j|^p \right)^{\frac{1}{p}} \left( \sum_{j=1}^{n-1} \|\Delta x_j\|^q \right)^{\frac{1}{q}} \sum_{1 \leq i < j \leq n} (j-i) p_i p_j, \end{aligned} \quad (5.87)$$

where  $p > 1$ ,  $\frac{1}{p} + \frac{1}{q} = 1$ .

The constant  $c = 1$  in the right hand side of (5.87) is sharp.

The case of equal weights is embodied in the following corollary [5].

**Corollary 248** With the above assumptions for  $\alpha_i, x_i$  ( $i = 1, \dots, n$ ) one has

$$\begin{aligned} 0 &\leq \left\| \frac{1}{n} \sum_{i=1}^n \alpha_i x_i - \frac{1}{n} \sum_{i=1}^n \alpha_i \cdot \frac{1}{n} \sum_{i=1}^n x_i \right\| \\ &\leq \frac{n^2 - 1}{6n} \left( \sum_{j=1}^{n-1} |\Delta \alpha_j|^p \right)^{\frac{1}{p}} \left( \sum_{j=1}^{n-1} \|\Delta x_j\|^q \right)^{\frac{1}{q}}, \end{aligned}$$

where  $p > 1$ ,  $\frac{1}{p} + \frac{1}{q} = 1$ .

The constant  $\frac{1}{6}$  is the best possible.

The main aim of this section is to establish some similar bounds for the absolute value of the difference

$$\sum_{i=1}^n p_i \langle x_i, y_i \rangle - \left\langle \sum_{i=1}^n p_i x_i, \sum_{i=1}^n p_i y_i \right\rangle$$

provided that  $x_i, y_i$  ( $i = 1, \dots, n$ ) are vectors in an inner product space  $H$ , and  $p_i \geq 0$  ( $i = 1, \dots, n$ ) with  $\sum_{i=1}^n p_i = 1$ .

### 5.5.2 The Main Results

We assume that  $(H, \langle \cdot, \cdot \rangle)$  is an inner product space over  $\mathbb{K}$ ,  $\mathbb{K} = \mathbb{C}$  or  $\mathbb{K} = \mathbb{R}$ . The following discrete inequality of Grüss' type holds.

**Theorem 249** *If  $x_i, y_i \in H$ ,  $p_i \geq 0$  ( $i = 1, \dots, n$ ) with  $\sum_{i=1}^n p_i = 1$ , then one has the inequalities:*

$$\begin{aligned} & \left| \sum_{i=1}^n p_i \langle x_i, y_i \rangle - \left\langle \sum_{i=1}^n p_i x_i, \sum_{i=1}^n p_i y_i \right\rangle \right| \\ & \leq \begin{cases} \left[ \sum_{i=1}^n i^2 p_i - \left( \sum_{i=1}^n i p_i \right)^2 \right] \max_{k=1, \dots, n-1} \|\Delta x_k\| \max_{k=1, \dots, n-1} \|\Delta y_k\|; \\ \left[ \sum_{1 \leq j < i \leq n} p_i p_j (i-j) \right] \left( \sum_{k=1}^{n-1} \|\Delta x_k\|^p \right)^{\frac{1}{p}} \left( \sum_{k=1}^{n-1} \|\Delta y_k\|^q \right)^{\frac{1}{q}} \\ \quad \text{if } p > 1, \frac{1}{p} + \frac{1}{q} = 1; \\ \frac{1}{2} \left[ \sum_{i=1}^n p_i (1-p_i) \right] \sum_{k=1}^{n-1} \|\Delta x_k\| \sum_{k=1}^{n-1} \|\Delta y_k\|. \end{cases} \end{aligned} \quad (5.88)$$

All the inequalities in (5.88) are sharp.

The following particular case for equal vectors holds.

**Corollary 250** *With the assumptions of Theorem 249, one has the inequalities*

$$\begin{aligned} 0 & \leq \sum_{i=1}^n p_i \|x_i\|^2 - \left\| \sum_{i=1}^n p_i x_i \right\|^2 \\ & \leq \begin{cases} \left[ \sum_{i=1}^n i^2 p_i - \left( \sum_{i=1}^n i p_i \right)^2 \right] \max_{k=1, \dots, n-1} \|\Delta x_k\|^2; \\ \sum_{1 \leq j < i \leq n} p_i p_j (i-j) \left( \sum_{k=1}^{n-1} \|\Delta x_k\|^p \right)^{\frac{1}{p}} \left( \sum_{k=1}^{n-1} \|\Delta x_k\|^q \right)^{\frac{1}{q}} \\ \quad \text{if } p > 1, \frac{1}{p} + \frac{1}{q} = 1; \\ \frac{1}{2} \sum_{i=1}^n p_i (1-p_i) \left( \sum_{k=1}^{n-1} \|\Delta x_k\| \right)^2. \end{cases} \end{aligned}$$

The following particular case for equal weights may be useful in practice.

**Corollary 251** *If  $x_i, y_i \in H$  ( $i = 1, \dots, n$ ), then one has the inequalities:*

$$\left| \frac{1}{n} \sum_{i=1}^n \langle x_i, y_i \rangle - \left\langle \frac{1}{n} \sum_{i=1}^n x_i, \frac{1}{n} \sum_{i=1}^n y_i \right\rangle \right|$$

$$\leq \begin{cases} \frac{n^2-1}{12} \max_{k=1,\dots,n-1} \|\Delta x_k\| \max_{k=1,\dots,n-1} \|\Delta y_k\|; \\ \frac{n^2-1}{6n} \left( \sum_{k=1}^{n-1} \|\Delta x_k\|^p \right)^{\frac{1}{p}} \left( \sum_{k=1}^{n-1} \|\Delta y_k\|^q \right)^{\frac{1}{q}} \\ \quad \text{if } p > 1, \frac{1}{p} + \frac{1}{q} = 1; \\ \frac{n-1}{2n} \sum_{k=1}^{n-1} \|\Delta x_k\| \sum_{k=1}^{n-1} \|\Delta y_k\|. \end{cases}$$

The constants  $\frac{1}{12}$ ,  $\frac{1}{6}$  and  $\frac{1}{2}$  are best possible.

In particular, the following corollary holds.

**Corollary 252** *If  $x_i \in H$  ( $i = 1, \dots, n$ ), then one has the inequality*

$$0 \leq \frac{1}{n} \sum_{i=1}^n \|x_i\|^2 - \left\| \frac{1}{n} \sum_{i=1}^n x_i \right\|^2 \leq \begin{cases} \frac{n^2-1}{12} \max_{k=1,n} \|\Delta x_k\|^2; \\ \frac{n^2-1}{6n} \left( \sum_{k=1}^{n-1} \|\Delta x_k\|^p \right)^{\frac{1}{p}} \left( \sum_{k=1}^{n-1} \|\Delta x_k\|^q \right)^{\frac{1}{q}} \\ \quad \text{if } p > 1, \frac{1}{p} + \frac{1}{q} = 1; \\ \frac{n-1}{2n} \left( \sum_{k=1}^{n-1} \|\Delta x_k\| \right)^2. \end{cases}$$

The constants  $\frac{1}{12}$ ,  $\frac{1}{6}$  and  $\frac{1}{2}$  are best possible.

### 5.5.3 Proof of the Main Result

It is well known that, the following identity holds in inner product spaces:

$$\begin{aligned} & \sum_{i=1}^n p_i \langle x_i, y_i \rangle - \left\langle \sum_{i=1}^n p_i x_i, \sum_{i=1}^n p_i y_i \right\rangle \\ &= \frac{1}{2} \sum_{i,j=1}^n p_i p_j \langle x_i - x_j, y_i - y_j \rangle \\ &= \sum_{1 \leq j < i \leq n} p_i p_j \langle x_i - x_j, y_i - y_j \rangle. \end{aligned} \tag{5.89}$$

We observe, for  $i > j$ , we can write that

$$x_i - x_j = \sum_{k=j}^{i-1} \Delta x_k, \quad y_i - y_j = \sum_{k=j}^{i-1} \Delta y_k. \tag{5.90}$$



Taking the modulus in (5.89) and by the use of (5.90) and Schwarz's inequality in inner product spaces, i.e., we recall that  $|\langle z, u \rangle| \leq \|z\| \|u\|$ ,  $z, u \in H$ , we have:

$$\begin{aligned}
& \left| \sum_{i=1}^n p_i \langle x_i, y_i \rangle - \left\langle \sum_{i=1}^n p_i x_i, \sum_{i=1}^n p_i y_i \right\rangle \right| \\
& \leq \sum_{1 \leq j < i \leq n} p_i p_j |\langle x_i - x_j, y_i - y_j \rangle| \\
& \leq \sum_{1 \leq j < i \leq n} p_i p_j \|x_i - x_j\| \|y_i - y_j\| \\
& = \sum_{1 \leq j < i \leq n} p_i p_j \left\| \sum_{k=j}^{i-1} \Delta x_k \right\| \left\| \sum_{l=j}^{i-1} \Delta y_l \right\| \\
& \leq \sum_{1 \leq j < i \leq n} p_i p_j \sum_{k=j}^{i-1} \|\Delta x_k\| \sum_{l=j}^{i-1} \|\Delta y_l\| \\
& := M.
\end{aligned}$$

It is obvious that

$$\sum_{k=j}^{i-1} \|\Delta x_k\| \leq (i-j) \max_{k=j, \dots, i-1} \|\Delta x_k\| \leq (i-j) \max_{k=1, \dots, n} \|\Delta x_k\|$$

and

$$\sum_{k=j}^{i-1} \|\Delta y_k\| \leq (i-j) \max_{k=j, \dots, i-1} \|\Delta y_k\| \leq (i-j) \max_{k=1, \dots, n} \|\Delta y_k\|,$$

giving that

$$M \leq \sum_{1 \leq j < i \leq n} p_i p_j (i-j)^2 \cdot \max_{k=1, \dots, n} \|\Delta x_k\| \max_{k=1, \dots, n} \|\Delta y_k\|,$$

and since

$$\sum_{1 \leq j < i \leq n} p_i p_j (i-j)^2 = \frac{1}{2} \sum_{i,j=1}^n p_i p_j (i-j)^2 = \sum_{i=1}^n p_i i^2 - \left( \sum_{i=1}^n i p_i \right)^2,$$

the first inequality in (5.88) is proved.

Using Hölder's discrete inequality, we can state that

$$\sum_{k=j}^{i-1} \|\Delta x_k\| \leq (i-j)^{\frac{1}{q}} \left( \sum_{k=j}^{i-1} \|\Delta x_k\|^p \right)^{\frac{1}{p}} \leq (i-j)^{\frac{1}{q}} \left( \sum_{k=1}^{n-1} \|\Delta x_k\|^p \right)^{\frac{1}{p}}$$

and

$$\sum_{k=j}^{i-1} \|\Delta y_k\| \leq (i-j)^{\frac{1}{p}} \left( \sum_{k=j}^{i-1} \|\Delta y_k\|^q \right)^{\frac{1}{q}} \leq (i-j)^{\frac{1}{p}} \left( \sum_{k=1}^{n-1} \|\Delta y_k\|^q \right)^{\frac{1}{q}},$$

for  $p > 1$ ,  $\frac{1}{p} + \frac{1}{q} = 1$ , giving that:

$$M \leq \sum_{1 \leq j < i \leq n} p_i p_j (i - j) \cdot \left( \sum_{k=1}^{n-1} \|\Delta x_k\|^p \right)^{\frac{1}{p}} \left( \sum_{k=1}^{n-1} \|\Delta y_k\|^q \right)^{\frac{1}{q}}$$

and the second inequality in (5.88) is proved.

Also, observe that

$$\sum_{k=j}^{i-1} \|\Delta x_k\| \leq \sum_{k=1}^{n-1} \|\Delta x_k\| \quad \text{and} \quad \sum_{k=j}^{i-1} \|\Delta y_k\| \leq \sum_{k=1}^{n-1} \|\Delta y_k\|$$

and thus

$$M \leq \sum_{1 \leq j < i \leq n} p_i p_j (i - j) \sum_{k=1}^{n-1} \|\Delta x_k\| \sum_{k=1}^{n-1} \|\Delta y_k\|.$$

Since

$$\begin{aligned} \sum_{1 \leq j < i \leq n} p_i p_j &= \frac{1}{2} \left[ \sum_{i,j=1}^n p_i p_j - \sum_{k=1}^n p_k^2 \right] \\ &= \frac{1}{2} \left( 1 - \sum_{k=1}^n p_k^2 \right) \\ &= \frac{1}{2} \sum_{i=1}^n p_i (1 - p_i), \end{aligned}$$

the last part of (5.88) is also proved.

Now, assume that the first inequality in (5.88) holds with a constant  $c > 0$ , i.e.,

$$\begin{aligned} &\sum_{i=1}^n p_i \langle x_i, y_i \rangle - \left\langle \sum_{i=1}^n p_i x_i, \sum_{i=1}^n p_i y_i \right\rangle \\ &\leq c \left[ \sum_{i=1}^n i^2 p_i - \left( \sum_{i=1}^n i p_i \right)^2 \right] \max_{k=1, \dots, n-1} \|\Delta x_k\| \max_{k=1, \dots, n-1} \|\Delta y_k\| \end{aligned}$$

and choose  $n = 2$  to get

$$p_1 p_2 |\langle x_2 - x_1, y_2 - y_1 \rangle| \leq c p_1 p_2 \|x_2 - x_1\| \|y_2 - y_1\| \quad (5.91)$$

for any  $p_1, p_2 > 0$  and  $x_1, x_2, y_1, y_2 \in H$ .

If in (5.91) we choose  $y_2 = x_2$ ,  $y_1 = x_1$  and  $x_2 \neq x_1$ , then we deduce  $c \geq 1$ , which proves the sharpness of the constant in the first inequality in (5.88).

In a similar way one may show that the other two inequalities are sharp, and the theorem is completely proved.

### 5.5.4 A Reverse for Jensen's Inequality

Let  $(H; \langle \cdot, \cdot \rangle)$  be a real inner product space and  $F : H \rightarrow \mathbb{R}$  a Fréchet differentiable convex function on  $H$ . If  $\nabla F : H \rightarrow H$  denotes the gradient operator associated to  $F$ , then we have the inequality

$$F(x) - F(y) \geq \langle \nabla F(y), x - y \rangle$$

for each  $x, y \in H$ .

The following result holds.

**Theorem 253** *Let  $F : H \rightarrow \mathbb{R}$  be as above and  $z_i \in H, i \in \{1, \dots, n\}$ . If  $q_i \geq 0$  ( $i \in \{1, \dots, n\}$ ) with  $\sum_{i=1}^n q_i = 1$ , then we have the following reverse of Jensen's inequality*

$$\begin{aligned} 0 &\leq \sum_{i=1}^n q_i F(z_i) - F\left(\sum_{i=1}^n q_i z_i\right) \\ &\leq \begin{cases} \left[ \sum_{i=1}^n i^2 q_i - \left(\sum_{i=1}^n i q_i\right)^2 \right] \max_{k=1, \dots, n-1} \|\Delta(\nabla F(z_i))\| \max_{k=1, \dots, n-1} \|\Delta z_i\|; \\ \left[ \sum_{1 \leq j < i \leq n} q_i q_j (i-j) \right] \left( \sum_{i=1}^{n-1} \|\Delta(\nabla F(z_i))\|^p \right)^{\frac{1}{p}} \left( \sum_{i=1}^{n-1} \|\Delta z_i\|^q \right)^{\frac{1}{q}} \\ \quad \text{if } p > 1, \frac{1}{p} + \frac{1}{q} = 1; \\ \frac{1}{2} \left[ \sum_{i=1}^n q_i (1 - q_i) \right] \sum_{i=1}^{n-1} \|\Delta(\nabla F(z_i))\| \sum_{i=1}^{n-1} \|\Delta z_i\|. \end{cases} \end{aligned} \quad (5.92)$$

**Proof.** We know, see for example [3, Eq. (4.4)], that the following reverse of Jensen's inequality for Fréchet differentiable convex functions

$$\begin{aligned} 0 &\leq \sum_{i=1}^n q_i F(z_i) - F\left(\sum_{i=1}^n q_i z_i\right) \\ &\leq \sum_{i=1}^n q_i \langle \nabla F(z_i), z_i \rangle - \left\langle \sum_{i=1}^n q_i \nabla F(z_i), \sum_{i=1}^n q_i z_i \right\rangle \end{aligned} \quad (5.93)$$

holds.

Now, if we apply Theorem 249 for the choices  $x_i = \nabla F(z_i), y_i = z_i$  and

$p_i = q_i$  ( $i = 1, \dots, n$ ), then we may state

$$\begin{aligned}
& \left| \sum_{i=1}^n q_i \langle \nabla F(z_i), z_i \rangle - \left\langle \sum_{i=1}^n q_i \nabla F(z_i), \sum_{i=1}^n q_i z_i \right\rangle \right| \\
& \leq \begin{cases} \left[ \sum_{i=1}^n i^2 q_i - \left( \sum_{i=1}^n i q_i \right)^2 \right] \max_{k=1, \dots, n-1} \|\Delta(\nabla F(z_k))\| \max_{k=1, \dots, n-1} \|\Delta z_k\| ; \\ \left[ \sum_{1 \leq j < i \leq n} q_i q_j (i-j) \right] \left( \sum_{k=1}^{n-1} \|\Delta(\nabla F(z_k))\|^p \right)^{\frac{1}{p}} \left( \sum_{k=1}^{n-1} \|\Delta z_k\|^q \right)^{\frac{1}{q}} \\ \quad \text{if } p > 1, \frac{1}{p} + \frac{1}{q} = 1; \\ \frac{1}{2} \left[ \sum_{i=1}^n q_i (1 - p_i) \right] \sum_{k=1}^{n-1} \|\Delta(\nabla F(z_k))\| \sum_{k=1}^{n-1} \|\Delta z_k\|. \end{cases} \tag{5.94}
\end{aligned}$$

Finally, on making use of the inequalities (5.93) and (5.94), we deduce the desired result (5.92). ■

The unweighted case may useful in application and is incorporated in the following corollary.

**Corollary 254** *Let  $F : H \rightarrow \mathbb{R}$  be as above and  $z_i \in H$ ,  $i \in \{1, \dots, n\}$ . Then we have the inequalities*

$$\begin{aligned}
0 & \leq \frac{1}{n} \sum_{i=1}^n F(z_i) - F\left(\frac{1}{n} \sum_{i=1}^n z_i\right) \\
& \leq \begin{cases} \frac{n^2 - 1}{12} \max_{k=1, \dots, n-1} \|\Delta(\nabla F(z_k))\| \max_{k=1, \dots, n-1} \|\Delta z_k\| ; \\ \frac{n^2 - 1}{6n} \left( \sum_{k=1}^{n-1} \|\Delta(\nabla F(z_k))\|^p \right)^{\frac{1}{p}} \left( \sum_{k=1}^{n-1} \|\Delta z_k\|^q \right)^{\frac{1}{q}} \\ \quad \text{if } p > 1, \frac{1}{p} + \frac{1}{q} = 1; \\ \frac{n-1}{2n} \sum_{k=1}^{n-1} \|\Delta(\nabla F(z_k))\| \sum_{k=1}^{n-1} \|\Delta z_k\|. \end{cases}
\end{aligned}$$

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## 5.6 Bounds for a Pair of n-Tuples of Vectors

### 5.6.1 Introduction

Let  $(H; \langle \cdot, \cdot \rangle)$  be an inner product over the real or complex number field  $\mathbb{K}$ . For  $\mathbf{p} = (p_1, \dots, p_n) \in \mathbb{R}^n$  and  $\mathbf{x} = (x_1, \dots, x_n)$ ,  $\mathbf{y} = (y_1, \dots, y_n) \in H^n$ , define the *Čebyšev functional*

$$T_n(\bar{\mathbf{p}}; \bar{\mathbf{x}}, \bar{\mathbf{y}}) := P_n \sum_{i=1}^n p_i \langle x_i, y_i \rangle - \left\langle \sum_{i=1}^n p_i x_i, \sum_{i=1}^n p_i y_i \right\rangle, \quad (5.95)$$

where  $P_n := \sum_{i=1}^n p_i$ .

The following Grüss type inequality has been obtained in [1].

**Theorem 255** *Let  $H$ ,  $\mathbf{x}, \mathbf{y}$  be as above and  $p_i \geq 0$  ( $i \in \{1, \dots, n\}$ ) with  $\sum_{i=1}^n p_i = 1$ , i.e.,  $\mathbf{p}$  is a probability sequence. If  $x, X, y, Y \in H$  are such that*

$$\operatorname{Re} \langle X - x_i, x_i - x \rangle \geq 0, \quad \operatorname{Re} \langle Y - y_i, y_i - y \rangle \geq 0 \quad (5.96)$$

for each  $i \in \{1, \dots, n\}$ , or, equivalently, (see [2])

$$\left\| x_i - \frac{x + X}{2} \right\| \leq \frac{1}{2} \|X - x\|, \quad \left\| y_i - \frac{y + Y}{2} \right\| \leq \frac{1}{2} \|Y - y\| \quad (5.97)$$

for each  $i \in \{1, \dots, n\}$ , then we have the inequality

$$|T_n(\bar{\mathbf{p}}; \bar{\mathbf{x}}, \bar{\mathbf{y}})| \leq \frac{1}{4} \|X - x\| \|Y - y\|. \quad (5.98)$$

The constant  $\frac{1}{4}$  is best possible in the sense that it cannot be replaced by a smaller constant.

In [3], the following Grüss type inequality for the forward difference of vectors was established.

**Theorem 256** *Let  $\mathbf{x} = (x_1, \dots, x_n)$ ,  $\mathbf{y} = (y_1, \dots, y_n) \in H^n$  and  $\mathbf{p} \in \mathbb{R}_+^n$  be a probability sequence. Then one has the inequality:*

$$|T_n(\bar{\mathbf{p}}; \bar{\mathbf{x}}, \bar{\mathbf{y}})| \leq \begin{cases} \left[ \sum_{i=1}^n i^2 p_i - \left( \sum_{i=1}^n i p_i \right)^2 \right] \max_{1 \leq k \leq n-1} \|\Delta x_k\| \max_{1 \leq k \leq n-1} \|\Delta y_k\|; \\ \sum_{1 \leq j < i \leq n} p_i p_j (i - j) \left( \sum_{k=1}^{n-1} \|\Delta x_k\|^p \right)^{\frac{1}{p}} \left( \sum_{k=1}^{n-1} \|\Delta y_k\|^q \right)^{\frac{1}{q}} \\ \quad \text{if } p > 1, \frac{1}{p} + \frac{1}{q} + 1 \\ \frac{1}{2} \left[ \sum_{i=1}^n p_i (1 - p_i) \right] \sum_{k=1}^{n-1} \|\Delta x_k\| \sum_{k=1}^{n-1} \|\Delta y_k\|. \end{cases} \quad (5.99)$$

The constants 1, 1 and  $\frac{1}{2}$  in the right hand side of inequality (5.99) are best in the sense that they cannot be replaced by smaller constants.

Another result is incorporated in the following theorem (see [2]).

**Theorem 257** *Let  $\mathbf{x}, \mathbf{y}$  and  $\mathbf{p}$  be as in Theorem 256. If there exist  $x, X \in H$  such that*

$$\operatorname{Re} \langle X - x_i, x_i - x \rangle \geq 0 \quad \text{for each } i \in \{1, \dots, n\}, \quad (5.100)$$

*or, equivalently,*

$$\left\| x_i - \frac{x + X}{2} \right\| \leq \frac{1}{2} \|X - x\| \quad \text{for each } i \in \{1, \dots, n\}, \quad (5.101)$$

*then one has the inequality*

$$\begin{aligned} |T_n(\bar{\mathbf{p}}; \bar{\mathbf{x}}, \bar{\mathbf{y}})| &\leq \frac{1}{2} \|X - x\| \left\| \sum_{i=1}^n p_i y_i - \sum_{j=1}^n p_j y_j \right\| \\ &\leq \frac{1}{2} \|X - x\| \left[ \sum_{i=1}^n p_i \|y_i\|^2 - \left\| \sum_{i=1}^n p_i y_i \right\|^2 \right]^{\frac{1}{2}}. \end{aligned} \quad (5.102)$$

*The constant  $\frac{1}{2}$  is best possible in the first and second inequalities in the sense that it cannot be replaced by a smaller constant.*

**Remark 258** *If  $\mathbf{x}$  and  $\mathbf{y}$  satisfy the assumptions of Theorem 255, then we have the following sequence of inequalities improving the Grüss inequality (5.98):*

$$\begin{aligned} |T_n(\bar{\mathbf{p}}; \bar{\mathbf{x}}, \bar{\mathbf{y}})| &\leq \frac{1}{2} \|X - x\| \left\| \sum_{i=1}^n p_i y_i - \sum_{j=1}^n p_j y_j \right\| \\ &\leq \frac{1}{2} \|X - x\| \left( \sum_{i=1}^n p_i \|y_i\|^2 - \left\| \sum_{i=1}^n p_i y_i \right\|^2 \right)^{\frac{1}{2}} \\ &\leq \frac{1}{4} \|X - x\| \|Y - y\|. \end{aligned} \quad (5.103)$$

Now, if we consider the Čebyšev functional for the uniform probability distribution  $u = (\frac{1}{n}, \dots, \frac{1}{n})$ ,

$$T_n(\bar{\mathbf{x}}, \bar{\mathbf{y}}) := \frac{1}{n} \sum_{i=1}^n \langle x_i, y_i \rangle - \left\langle \frac{1}{n} \sum_{i=1}^n x_i, \frac{1}{n} \sum_{i=1}^n y_i \right\rangle,$$

then, with the assumptions of Theorem 255, we have

$$|T_n(\bar{\mathbf{x}}, \bar{\mathbf{y}})| \leq \frac{1}{4} \|X - x\| \|Y - y\|. \quad (5.104)$$

Theorem 256 will provide the following inequalities

$$\begin{aligned}
& |T_n(\bar{\mathbf{x}}, \bar{\mathbf{y}})| \\
& \leq \begin{cases} \frac{1}{12} (n^2 - 1) \max_{1 \leq k \leq n-1} \|\Delta x_k\| \max_{1 \leq k \leq n-1} \|\Delta y_k\|; \\ \frac{1}{6} \left(n - \frac{1}{n}\right) \left(\sum_{k=1}^{n-1} \|\Delta x_k\|^p\right)^{\frac{1}{p}} \left(\sum_{k=1}^{n-1} \|\Delta y_k\|^q\right)^{\frac{1}{q}} & \text{if } p > 1, \frac{1}{p} + \frac{1}{q} + 1; \\ \frac{1}{2} \left(1 - \frac{1}{n}\right) \sum_{k=1}^{n-1} \|\Delta x_k\| \sum_{k=1}^{n-1} \|\Delta y_k\|. \end{cases}
\end{aligned} \tag{5.105}$$

Here the constants  $\frac{1}{12}$ ,  $\frac{1}{6}$  and  $\frac{1}{2}$  are best possible in the above sense.

Finally, from (5.103), we have

$$\begin{aligned}
|T_n(\bar{\mathbf{x}}, \bar{\mathbf{y}})| & \leq \frac{1}{2n} \|X - x\| \left\| \sum_{i=1}^n y_i - \frac{1}{n} \sum_{j=1}^n y_j \right\| \\
& \leq \frac{1}{2} \|X - x\| \left( \frac{1}{n} \sum_{i=1}^n \|y_i\|^2 - \left\| \frac{1}{n} \sum_{i=1}^n y_i \right\|^2 \right)^{\frac{1}{2}} \\
& \leq \frac{1}{4} \|X - x\| \|Y - y\|.
\end{aligned} \tag{5.106}$$

It is the main aim of this section to point out other bounds for the Čebyšev functionals  $T_n(\mathbf{p}, \mathbf{x}, \mathbf{y})$  and  $T_n(\mathbf{x}, \mathbf{y})$ .

### 5.6.2 Identities for Inner Products

For  $\mathbf{p} = (p_1, \dots, p_n) \in \mathbb{R}^n$  and  $\mathbf{a} = (a_1, \dots, a_n) \in H^n$  we define

$$P_i := \sum_{k=1}^i p_k, \quad \bar{P}_i = P_n - P_i, \quad i \in \{1, \dots, n-1\}$$

and the vectors

$$A_i(\mathbf{p}) = \sum_{k=1}^i p_k a_k, \quad \bar{A}_i(\mathbf{p}) = A_n(\mathbf{p}) - A_i(\mathbf{p})$$

for  $i \in \{1, \dots, n-1\}$ .

The following result holds.



**Theorem 259** Let  $(H; \langle \cdot, \cdot \rangle)$  be an inner product space over  $\mathbb{K}$ ,  $\mathbf{p} = (p_1, \dots, p_n) \in \mathbb{R}^n$  and  $\mathbf{a} = (a_1, \dots, a_n)$ ,  $\mathbf{b} = (b_1, \dots, b_n) \in H^n$ . Then we have the identities

$$\begin{aligned}
T_n(\mathbf{p}; \mathbf{a}, \mathbf{b}) &= \sum_{i=1}^{n-1} \langle P_i A_n(\mathbf{p}) - P_n A_i(\mathbf{p}), \Delta b_i \rangle \\
&= P_n \sum_{i=1}^{n-1} P_i \left\langle \frac{1}{P_n} A_n(\mathbf{p}) - \frac{1}{P_i} A_i(\mathbf{p}), \Delta b_i \right\rangle \\
&\quad (\text{if } P_i \neq 0, i \in \{1, \dots, n\}) \\
&= \sum_{i=1}^{n-1} P_i \bar{P}_i \left\langle \frac{1}{\bar{P}_i} \bar{A}_i(\mathbf{p}) - \frac{1}{P_i} A_i(\mathbf{p}), \Delta b_i \right\rangle \\
&\quad (\text{if } P_i, \bar{P}_i \neq 0, i \in \{1, \dots, n-1\}),
\end{aligned} \tag{5.107}$$

where  $\Delta x_i = x_{i+1} - x_i$  ( $i \in \{1, \dots, n-1\}$ ) is the forward difference.

**Proof.** We use the following summation by parts formula for vectors in inner product spaces

$$\sum_{l=p}^{q-1} \langle d_l, \Delta v_l \rangle = \langle d_l, v_l \rangle \Big|_p^q - \sum_{l=p}^{q-1} \langle v_{l+1}, \Delta d_l \rangle \tag{5.108}$$

where  $d_l, v_l$  are vectors in  $H$ ,  $l = p, \dots, q$  ( $q > p$ ;  $p, q$  are natural numbers).

If we choose in (5.108),  $p = 1$ ,  $q = n$ ,  $d_i = P_i A_n(\mathbf{p}) - P_n A_i(\mathbf{p})$  and  $v_i = b_i$  ( $i \in \{1, \dots, n-1\}$ ), then we get

$$\begin{aligned}
&\sum_{i=1}^{n-1} \langle P_i A_n(\mathbf{p}) - P_n A_i(\mathbf{p}), \Delta b_i \rangle \\
&= \langle P_i A_n(\mathbf{p}) - P_n A_i(\mathbf{p}), b_i \rangle \Big|_1^n - \sum_{i=1}^{n-1} \langle \Delta(P_i A_n(\mathbf{p}) - P_n A_i(\mathbf{p})), b_{i+1} \rangle \\
&= \langle P_n A_n(\mathbf{p}) - P_n A_n(\mathbf{p}), b_n \rangle - \langle P_1 A_n(\mathbf{p}) - P_n A_1(\mathbf{p}), b_1 \rangle \\
&\quad - \sum_{i=1}^{n-1} \langle P_{i+1} A_n(\mathbf{p}) - P_n A_{i+1}(\mathbf{p}) - P_i A_n(\mathbf{p}) + P_n A_i(\mathbf{p}), b_{i+1} \rangle \\
&= P_n p_1 \langle a_1, x_1 \rangle - p_1 \langle A_n(\mathbf{p}), b_1 \rangle - \left\langle A_n(\mathbf{p}), \sum_{i=1}^{n-1} p_{i+1} b_{i+1} \right\rangle \\
&\quad + P_n \sum_{i=1}^{n-1} p_{i+1} \langle a_{i+1}, b_{i+1} \rangle \\
&= P_n \sum_{i=1}^n p_i \langle a_i, b_i \rangle - \left\langle \sum_{i=1}^n p_i a_i, \sum_{i=1}^n p_i b_i \right\rangle \\
&= T_n(\mathbf{p}; \mathbf{a}, \mathbf{b}),
\end{aligned}$$

proving the first identity in (5.107).

The second and third identities are obvious and we omit the details. ■

The following lemma is of interest in itself.

**Lemma 260** *Let  $\mathbf{p} = (p_1, \dots, p_n) \in \mathbb{R}^n$  and  $\mathbf{a} = (a_1, \dots, a_n) \in H$ . Then we have the equality*

$$P_i A_n(\mathbf{p}) - P_n A_i(\mathbf{p}) = \sum_{j=1}^{n-1} P_{\min\{i,j\}} \bar{P}_{\max\{i,j\}} \Delta a_j \quad (5.109)$$

for each  $i \in \{1, \dots, n-1\}$ .

**Proof.** Define, for  $i \in \{1, \dots, n-1\}$ , the vector

$$K(i) := \sum_{j=1}^{n-1} P_{\min\{i,j\}} \bar{P}_{\max\{i,j\}} \cdot \Delta a_j.$$

We have

$$\begin{aligned} K(i) &= \sum_{j=1}^i P_{\min\{i,j\}} \bar{P}_{\max\{i,j\}} \cdot \Delta a_j + \sum_{j=i+1}^{n-1} P_{\min\{i,j\}} \bar{P}_{\max\{i,j\}} \cdot \Delta a_j \quad (5.110) \\ &= \sum_{j=1}^i P_j \bar{P}_i \cdot \Delta a_j + \sum_{j=i+1}^{n-1} P_i \bar{P}_j \cdot \Delta a_j \\ &= \bar{P}_i \sum_{j=1}^i P_j \cdot \Delta a_j + P_i \sum_{j=i+1}^{n-1} \bar{P}_j \cdot \Delta a_j. \end{aligned}$$

Using the summation by parts formula, we have

$$\begin{aligned} \sum_{j=1}^i P_j \cdot \Delta a_j &= P_j a_j \Big|_1^{i+1} - \sum_{j=1}^i (P_{j+1} - P_j) a_{j+1} \quad (5.111) \\ &= P_{i+1} a_{i+1} - p_1 a_1 - \sum_{j=1}^i p_{j+1} a_{j+1} \\ &= P_{i+1} a_{i+1} - \sum_{j=1}^{i+1} p_j a_j \end{aligned}$$

and

$$\begin{aligned} \sum_{j=i+1}^{n-1} \bar{P}_j \cdot \Delta a_j &= \bar{P}_j a_j \Big|_{i+1}^n - \sum_{j=i+1}^{n-1} (\bar{P}_{j+1} - \bar{P}_j) a_{j+1} \quad (5.112) \\ &= \bar{P}_n a_n - \bar{P}_{i+1} a_{i+1} - \sum_{j=i+1}^{n-1} (P_n - P_{j+1} - P_n + P_j) a_{j+1} \\ &= -\bar{P}_{i+1} a_{i+1} + \sum_{j=i+1}^{n-1} p_{j+1} a_{j+1}. \end{aligned}$$

Using (5.111) and (5.112), we have

$$\begin{aligned}
K(i) &= \bar{P}_i \left( P_{i+1} a_{i+1} - \sum_{j=1}^{i+1} p_j a_j \right) + P_i \left( \sum_{j=i+1}^{n-1} p_{j+1} a_{j+1} - \bar{P}_{i+1} a_{i+1} \right) \\
&= \bar{P}_i P_{i+1} a_{i+1} - \bar{P}_i \bar{P}_{i+1} a_{i+1} - \bar{P}_i \sum_{j=1}^{i+1} p_j a_j + P_i \sum_{j=i+1}^{n-1} p_{j+1} a_{j+1} \\
&= [(P_n - P_i) P_{i+1} - P_i (P_n - P_{i+1})] a_{i+1} + P_i \sum_{j=i+1}^{n-1} p_{j+1} a_{j+1} - \bar{P}_i \sum_{j=1}^{i+1} p_j a_j \\
&= P_n p_{i+1} a_{i+1} + P_i \sum_{j=i+1}^{n-1} p_{j+1} a_{j+1} - \bar{P}_i \sum_{j=1}^{i+1} p_j a_j \\
&= (P_i + \bar{P}_i) p_{i+1} a_{i+1} + P_i \sum_{j=i+1}^{n-1} p_{j+1} a_{j+1} - \bar{P}_i \sum_{j=1}^{i+1} p_j a_j \\
&= P_i \sum_{j=i+1}^{n-1} p_j a_j - \bar{P}_i \sum_{j=1}^i p_j a_j \\
&= P_i \bar{A}_i(\mathbf{p}) - \bar{P}_i A_i(\mathbf{p}) \\
&= P_i A_n(\mathbf{p}) - P_n A_i(\mathbf{p}),
\end{aligned}$$

and the identity is proved. ■

We are able now to state and prove the second identity for the Čebyšev functional.

**Theorem 261** *With the assumptions of Theorem 259, we have the identity*

$$T_n(\mathbf{p}; \mathbf{a}, \mathbf{b}) = \sum_{i=1}^{n-1} \sum_{j=1}^{n-1} P_{\min\{i,j\}} \bar{P}_{\max\{i,j\}} \cdot \langle \Delta a_j, \Delta b_i \rangle. \quad (5.113)$$

**Proof.** Follows by Theorem 259 and Lemma 260 and we omit the details. ■

### 5.6.3 New Inequalities

The following result holds.

**Theorem 262** *Let  $(H; \langle \cdot, \cdot \rangle)$  be an inner product space over the real or complex number field  $\mathbb{K}$ ;  $\mathbf{p} = (p_1, \dots, p_n) \in \mathbb{R}^n$  and  $\mathbf{a} = (a_1, \dots, a_n), \mathbf{b} = (b_1, \dots, b_n) \in$*

$H^n$ . Then we have the inequalities

$$|T_n(\mathbf{p}; \mathbf{a}, \mathbf{b})| \leq \begin{cases} \max_{1 \leq i \leq n-1} \|P_i A_n(\mathbf{p}) - P_n A_i(\mathbf{p})\| \sum_{j=1}^{n-1} \|\Delta b_j\|; \\ \left( \sum_{i=1}^{n-1} \|P_i A_n(\mathbf{p}) - P_n A_i(\mathbf{p})\|^q \right)^{\frac{1}{q}} \left( \sum_{j=1}^{n-1} \|\Delta b_j\|^p \right)^{\frac{1}{p}} \\ \text{for } p > 1, \frac{1}{p} + \frac{1}{q} = 1; \\ \sum_{i=1}^{n-1} \|P_i A_n(\mathbf{p}) - P_n A_i(\mathbf{p})\| \cdot \max_{1 \leq j \leq n-1} \|\Delta b_j\|. \end{cases} \quad (5.114)$$

All the inequalities in (5.114) are sharp in the sense that the constants 1 cannot be replaced by smaller constants.

**Proof.** Using the first identity in (5.107) and Schwarz's inequality in  $H$ , i.e.,  $|\langle u, v \rangle| \leq \|u\| \|v\|$ ,  $u, v \in H$ , we have successively:

$$\begin{aligned} |T_n(\mathbf{p}; \mathbf{a}, \mathbf{b})| &\leq \sum_{i=1}^{n-1} |\langle P_i A_n(\mathbf{p}) - P_n A_i(\mathbf{p}), \Delta b_i \rangle| \\ &\leq \sum_{i=1}^{n-1} \|P_i A_n(\mathbf{p}) - P_n A_i(\mathbf{p})\| \|\Delta b_i\|. \end{aligned}$$

Using Hölder's inequality, we deduce the desired result (5.114).

Let us prove, for instance, that the constant 1 in the second inequality is best possible.

Assume, for  $c > 0$ , we have that

$$|T_n(\mathbf{p}; \mathbf{a}, \mathbf{b})| \leq c \left( \sum_{i=1}^{n-1} \|P_i A_n(\mathbf{p}) - P_n A_i(\mathbf{p})\|^q \right)^{\frac{1}{q}} \left( \sum_{j=1}^{n-1} \|\Delta b_j\|^p \right)^{\frac{1}{p}} \quad (5.115)$$

for  $p > 1$ ,  $\frac{1}{p} + \frac{1}{q} = 1$ ,  $n \geq 2$ .

If we choose  $n = 2$ , then we get

$$|T_2(\mathbf{p}; \mathbf{a}, \mathbf{b})| \leq p_1 p_2 \langle a_2 - a_1, b_2 - b_1 \rangle.$$

Also, for  $n = 2$ ,

$$\left( \sum_{i=1}^{n-1} \|P_i A_n(\mathbf{p}) - P_n A_i(\mathbf{p})\|^q \right)^{\frac{1}{q}} = |p_1 p_2| \|a_2 - a_1\|$$

and

$$\left( \sum_{j=1}^{n-1} \|\Delta b_j\|^p \right)^{\frac{1}{p}} = \|b_2 - b_1\|,$$

and then, from (5.115), for  $n = 2$ , we deduce

$$|p_1 p_2| |\langle a_2 - a_1, b_2 - b_1 \rangle| \leq c |p_1 p_2| \|a_2 - a_1\| \|b_2 - b_1\|. \quad (5.116)$$

If in (5.116) we choose  $a_2 = b_2$ ,  $a_2 = b_1$  and  $b_2 \neq b_1$ ,  $p_1, p_2 \neq 0$ , we deduce  $c \geq 1$ , proving that 1 is the best possible constant in that inequality. ■

The following corollary for the uniform distribution of the probability  $\mathbf{p}$  holds.

**Corollary 263** *With the assumptions of Theorem 262 for  $\mathbf{a}$  and  $\mathbf{b}$ , we have the inequalities*

$$0 \leq |T_n(\mathbf{a}, \mathbf{b})| \leq \frac{1}{n^2} \begin{cases} \max_{1 \leq i \leq n-1} \left\| i \sum_{k=1}^n a_k - n \sum_{k=1}^i a_k \right\| \left\| \sum_{j=1}^{n-1} \Delta b_j \right\|; \\ \left( \sum_{i=1}^{n-1} \left\| i \sum_{k=1}^n a_k - n \sum_{k=1}^i a_k \right\|^q \right)^{\frac{1}{q}} \left( \sum_{j=1}^{n-1} \|\Delta b_j\|^p \right)^{\frac{1}{p}} \\ \quad \text{for } p > 1, \frac{1}{p} + \frac{1}{q} = 1; \\ \sum_{i=1}^{n-1} \left\| i \sum_{k=1}^n a_k - n \sum_{k=1}^i a_k \right\| \cdot \max_{1 \leq j \leq n-1} \|\Delta b_j\|. \end{cases} \quad (5.117)$$

The following result may be stated as well.

**Theorem 264** *With the assumptions of Theorem 262 and if  $P_i \neq 0$  ( $i = 1, \dots, n$ ), then we have the inequalities*

$$|T_n(\mathbf{p}; \mathbf{a}, \mathbf{b})| \leq |P_n| \times \begin{cases} \max_{1 \leq i \leq n-1} \left\| \frac{1}{P_n} A_n(\mathbf{p}) - \frac{1}{P_i} A_i(\mathbf{p}) \right\| \sum_{i=1}^{n-1} |P_i| \|\Delta b_i\|; \\ \left( \sum_{i=1}^{n-1} |P_i| \left\| \frac{1}{P_n} A_n(\mathbf{p}) - \frac{1}{P_i} A_i(\mathbf{p}) \right\|^q \right)^{\frac{1}{q}} \left( \sum_{i=1}^n |P_i| \|\Delta b_i\|^p \right)^{\frac{1}{p}} \\ \quad \text{for } p > 1, \frac{1}{p} + \frac{1}{q} = 1; \\ \sum_{i=1}^{n-1} |P_i| \left\| \frac{1}{P_n} A_n(\mathbf{p}) - \frac{1}{P_i} A_i(\mathbf{p}) \right\| \cdot \max_{1 \leq i \leq n-1} \|\Delta b_i\|. \end{cases} \quad (5.118)$$

All the inequalities in (5.118) are sharp in the sense that the constant 1 cannot be replaced by a smaller constant.

**Proof.** Using the second equality in (5.107) and Schwarz's inequality, we have

$$\begin{aligned} |T_n(\mathbf{p}; \mathbf{a}, \mathbf{b})| &\leq |P_n| \sum_{i=1}^{n-1} |P_i| \left| \left\langle \frac{1}{P_n} A_n(\mathbf{p}) - \frac{1}{P_i} A_i(\mathbf{p}), \Delta b_i \right\rangle \right| \\ &\leq |P_n| \sum_{i=1}^{n-1} |P_i| \left\| \frac{1}{P_n} A_n(\mathbf{p}) - \frac{1}{P_i} A_i(\mathbf{p}) \right\| \|\Delta b_i\|. \end{aligned}$$

Using Hölder's weighted inequality, we deduce (5.118).

The sharpness of the constant may be proven in a similar manner to the one in Theorem 262. We omit the details. ■

The following corollary containing the unweighted inequalities holds.

**Corollary 265** *With the above assumptions for  $\mathbf{a}$  and  $\mathbf{b}$ , one has*

$$|T_n(\mathbf{a}, \mathbf{b})| \leq \frac{1}{n} \begin{cases} \max_{1 \leq i \leq n-1} \left\| \frac{1}{n} \sum_{k=1}^n a_k - \frac{1}{i} \sum_{k=1}^i a_k \right\| \sum_{i=1}^{n-1} i \|\Delta b_i\|; \\ \left( \sum_{i=1}^{n-1} i \left\| \frac{1}{n} \sum_{k=1}^n a_k - \frac{1}{i} \sum_{k=1}^i a_k \right\|^q \right)^{\frac{1}{q}} \left( \sum_{i=1}^{n-1} i \|\Delta b_i\|^p \right)^{\frac{1}{p}} \\ \text{for } p > 1, \frac{1}{p} + \frac{1}{q} = 1; \\ \sum_{i=1}^{n-1} i \left\| \frac{1}{n} \sum_{k=1}^n a_k - \frac{1}{i} \sum_{k=1}^i a_k \right\| \cdot \max_{1 \leq i \leq n-1} \|\Delta b_i\|. \end{cases} \quad (5.119)$$

The inequalities (5.119) are sharp in the sense mentioned above.

Another type of inequality may be stated if ones used the third identity in (5.107) and Hölder's weighted inequality with the weights:  $|P_i| |\bar{P}_i|$ ,  $i \in \{1, \dots, n-1\}$ .

**Theorem 266** *With the assumptions in Theorem 262 and if  $P_i, \bar{P}_i \neq 0$ ,  $i \in \{1, \dots, n-1\}$ , then we have the inequalities*

$$\begin{aligned} &|T_n(\mathbf{p}; \mathbf{a}, \mathbf{b})| \\ &\leq |P_n| \times \begin{cases} \max_{1 \leq i \leq n-1} \left\| \frac{1}{\bar{P}_i} \bar{A}_i(\mathbf{p}) - \frac{1}{P_i} A_i(\mathbf{p}) \right\| \sum_{i=1}^{n-1} |P_i| |\bar{P}_i| \|\Delta b_i\|; \\ \left( \sum_{i=1}^{n-1} |P_i| |\bar{P}_i| \left\| \frac{1}{\bar{P}_i} \bar{A}_i(\mathbf{p}) - \frac{1}{P_i} A_i(\mathbf{p}) \right\|^q \right)^{\frac{1}{q}} \left( \sum_{i=1}^{n-1} |P_i| |\bar{P}_i| \|\Delta b_i\|^p \right)^{\frac{1}{p}} \\ \text{for } p > 1, \frac{1}{p} + \frac{1}{q} = 1; \\ \sum_{i=1}^{n-1} |P_i| |\bar{P}_i| \left\| \frac{1}{\bar{P}_i} \bar{A}_i(\mathbf{p}) - \frac{1}{P_i} A_i(\mathbf{p}) \right\| \cdot \max_{1 \leq i \leq n-1} \|\Delta b_i\|. \end{cases} \end{aligned} \quad (5.120)$$

In particular, if  $p_i = \frac{1}{n}$ ,  $i \in \{1, \dots, n\}$ , then we have

$$\begin{aligned}
& |T_n(\mathbf{a}, \mathbf{b})| \\
& \leq \frac{1}{n^2} \left\{ \begin{aligned} & \max_{1 \leq i \leq n-1} \left\| \frac{1}{n-i} \sum_{k=i+1}^n a_k - \frac{1}{i} \sum_{k=1}^i a_k \right\| \sum_{i=1}^{n-1} i(n-i) \|\Delta b_i\| ; \\ & \left( \sum_{i=1}^{n-1} i(n-i) \left\| \frac{1}{n-i} \sum_{k=i+1}^n a_k - \frac{1}{i} \sum_{k=1}^i a_k \right\|^q \right)^{\frac{1}{q}} \left( \sum_{i=1}^{n-1} i(n-i) \|\Delta b_i\|^p \right)^{\frac{1}{p}} \\ & \text{for } p > 1, \frac{1}{p} + \frac{1}{q} = 1; \\ & \sum_{i=1}^{n-1} i(n-i) \left\| \frac{1}{n-i} \sum_{k=i+1}^n a_k - \frac{1}{i} \sum_{k=1}^i a_k \right\| \cdot \max_{1 \leq i \leq n-1} \|\Delta b_i\|. \end{aligned} \right. \quad (5.121)
\end{aligned}$$

The inequalities in (5.120) and (5.121) are sharp in the above mentioned sense.

A different approach may be considered if one uses the representation in terms of double sums for the Čebyšev functional provided by Theorem 261.

The following result holds.

**Theorem 267** *With the above assumptions of Theorem 262, we have the inequalities*

$$\begin{aligned}
& |T_n(\mathbf{p}; \mathbf{a}, \mathbf{b})| \\
& \leq |P_n| \times \left\{ \begin{aligned} & \max_{1 \leq i, j \leq n-1} \{ |P_{\min\{i, j\}}|, |\bar{P}_{\max\{i, j\}}| \} \sum_{i=1}^{n-1} \|\Delta a_i\| \sum_{i=1}^{n-1} \|\Delta b_i\| ; \\ & \left( \sum_{i=1}^{n-1} \sum_{j=1}^{n-1} |P_{\min\{i, j\}}|^q |\bar{P}_{\max\{i, j\}}|^q \right)^{\frac{1}{q}} \left( \sum_{i=1}^{n-1} \|\Delta a_i\|^p \right)^{\frac{1}{p}} \left( \sum_{i=1}^{n-1} \|\Delta b_i\|^p \right)^{\frac{1}{p}} \\ & \text{for } p > 1, \frac{1}{p} + \frac{1}{q} = 1; \\ & \sum_{i=1}^{n-1} \sum_{j=1}^{n-1} |P_{\min\{i, j\}}| |\bar{P}_{\max\{i, j\}}| \max_{1 \leq i \leq n-1} \|\Delta a_i\| \max_{1 \leq i \leq n-1} \|\Delta b_i\|. \end{aligned} \right. \quad (5.122)
\end{aligned}$$

The inequalities are sharp in the sense mentioned above.

The proof follows by the identity (5.113) on using Hölder's inequality for double sums and we omit the details.

Now, define

$$k_\infty := \max_{1 \leq i, j \leq n-1} \left\{ \frac{\min\{i, j\}}{n} \left( 1 - \frac{\max\{i, j\}}{n} \right) \right\}, \quad n \geq 2.$$

Using the elementary inequality

$$ab \leq \frac{1}{4}(a+b)^2, \quad a, b \in \mathbb{R};$$

we deduce

$$\min\{i, j\} (n - \max\{i, j\}) \leq \frac{1}{4} (n - |i - j|)^2$$

for  $1 \leq i, j \leq n - 2$ . Consequently, we deduce

$$k_\infty \leq \frac{1}{4n^2} \max_{1 \leq i, j \leq n-1} \left\{ (n - |i - j|)^2 \right\} = \frac{1}{4}.$$

We may now state the following corollary of Theorem 267.

**Corollary 268** *With the assumptions of Theorem 262 for  $\mathbf{a}$  and  $\mathbf{b}$ , we have the inequality*

$$\begin{aligned} |T_n(\mathbf{a}, \mathbf{b})| &\leq k_\infty \sum_{i=1}^{n-1} \|\Delta a_i\| \sum_{i=1}^{n-1} \|\Delta b_i\| \\ &\leq \frac{1}{4} \sum_{i=1}^{n-1} \|\Delta a_i\| \sum_{i=1}^{n-1} \|\Delta b_i\|. \end{aligned} \quad (5.123)$$

The constant  $\frac{1}{4}$  cannot be replaced in general by a smaller constant.

**Remark 269** *The inequality (5.123) is better than the third inequality in (5.105).*

Consider now, for  $q > 1$ , the number

$$k_q := \frac{1}{n^2} \left( \sum_{i=1}^{n-1} \sum_{j=1}^{n-1} [\min\{i, j\} (n - \max\{i, j\})]^q \right)^{\frac{1}{q}}.$$

We observe, by symmetry of the terms under the summation symbol, we have that

$$k_q = \frac{1}{n^2} \left( 2 \sum_{1 \leq i < j \leq n-1} i^q (n - j)^q + \sum_{i=1}^{n-1} i^q (n - i)^q \right)^{\frac{1}{q}},$$

that may be computed exactly if  $q = 2$  or another natural number.

Since, as above,

$$[\min\{i, j\} (n - \max\{i, j\})]^q \leq \frac{1}{4^q} (n - |i - j|)^{2q},$$

we deduce

$$\begin{aligned} k_q &\leq \frac{1}{4n^2} \left( \sum_{i=1}^{n-1} \sum_{j=1}^{n-1} (n - |i - j|)^{2q} \right)^{\frac{1}{q}} \\ &\leq \frac{1}{4n^2} \left[ (n - 1)^2 n^{2q} \right]^{\frac{1}{q}} \\ &= \frac{1}{4} (n - 1)^{\frac{2}{q}}. \end{aligned}$$



Consequently, we may state the following corollary as well.

**Corollary 270** *With the assumptions of Theorem 262 for  $\mathbf{a}$  and  $\mathbf{b}$ , we have the inequalities*

$$\begin{aligned} |T_n(\mathbf{a}, \mathbf{b})| &\leq k_q \left( \sum_{i=1}^{n-1} \|\Delta a_i\|^p \right)^{\frac{1}{p}} \left( \sum_{i=1}^{n-1} \|\Delta b_i\|^p \right)^{\frac{1}{p}} \\ &\leq \frac{1}{4} (n-1)^{\frac{2}{q}} \left( \sum_{i=1}^{n-1} \|\Delta a_i\|^p \right)^{\frac{1}{p}} \left( \sum_{i=1}^{n-1} \|\Delta b_i\|^p \right)^{\frac{1}{p}}, \end{aligned} \quad (5.124)$$

provided  $p > 1$ ,  $\frac{1}{p} + \frac{1}{q} = 1$ . The constant  $\frac{1}{4}$  cannot be replaced in general by a smaller constant.

Finally, if we denote

$$k_1 := \frac{1}{n^2} \sum_{i=1}^{n-1} \sum_{j=1}^{n-1} [\min\{i, j\} (n - \max\{i, j\})],$$

then we observe, for  $\mathbf{u} = (\frac{1}{n}, \dots, \frac{1}{n})$ ,  $\mathbf{e} = (1, 2, \dots, n)$ , that

$$k_1 = |T_n(\mathbf{u}; \mathbf{e}, \mathbf{e})| = \frac{1}{n} \sum_{i=1}^n i^2 - \left( \frac{1}{n} \sum_{i=1}^n i \right)^2 = \frac{1}{12} (n^2 - 1),$$

and by Theorem 267, we deduce the inequality

$$|T_n(\mathbf{a}, \mathbf{b})| \leq \frac{1}{12} (n^2 - 1) \max_{1 \leq j \leq n-1} \|\Delta a_j\| \max_{1 \leq j \leq n-1} \|\Delta b_j\|.$$

Note that, the above inequality has been discovered using a different method in [3]. The constant  $\frac{1}{12}$  is best possible.

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## Chapter 6

# Other Inequalities in Inner Product Spaces

### 6.1 The Ostrowski Inequality

#### 6.1.1 Introduction

In 1951, A.M. Ostrowski [3, p. 289] proved the following result (see also [2, p. 92]):

**Theorem 271** *Suppose that  $\mathbf{a}, \mathbf{b}$  and  $\mathbf{x}$  are real  $n$ -tuples such that  $\mathbf{a} \neq 0$  and*

$$\sum_{i=1}^n a_i x_i = 0 \text{ and } \sum_{i=1}^n b_i x_i = 1.$$

*Then*

$$\sum_{i=1}^n x_i^2 \geq \frac{\sum_{i=1}^n a_i^2}{\sum_{i=1}^n a_i^2 \sum_{i=1}^n b_i^2 - (\sum_{i=1}^n a_i b_i)^2}$$

*with equality if and only if*

$$x_k = \frac{b_k \sum_{i=1}^n a_i^2 - a_k \sum_{i=1}^n a_i b_i}{\sum_{i=1}^n a_i^2 \sum_{i=1}^n b_i^2 - (\sum_{i=1}^n a_i b_i)^2},$$

*for  $k \in \{1, \dots, n\}$ .*

An integral version of this inequality was obtained by Pearce, Pečarić and Varošanec in 1998, [4].

H. Šikić and T. Šikić in 2001, [5], by the use of an argument based on orthogonal projections in inner product spaces have observed that Ostrowski's inequality may be naturally stated in an abstract setting as follows:

**Theorem 272** Let  $(H; \langle \cdot, \cdot \rangle)$  be a real or complex inner product space and  $a, b \in H$  two linearly independent vectors. If  $x \in H$  is such that

$$\langle x, a \rangle = 0 \text{ and } \langle x, b \rangle = 1,$$

then one has the inequality

$$\|x\|^2 \geq \frac{\|a\|^2}{\|a\|^2 \|b\|^2 - |\langle a, b \rangle|^2}, \quad (6.1)$$

with equality if and only if

$$x = \frac{\|a\|^2 b - \overline{\langle a, b \rangle} \cdot a}{\|a\|^2 \|b\|^2 - |\langle a, b \rangle|^2}.$$

In the present section, by the use of elementary arguments and Schwarz's inequality in inner product spaces, we show that Ostrowski's inequality (6.1) holds true for a larger class of elements  $x \in H$ . The case of equality is analyzed. Applications for complex sequences and integrals are also provided.

### 6.1.2 The General Inequality

The following theorem holds [1].

**Theorem 273** Let  $(H; \langle \cdot, \cdot \rangle)$  be a real or complex inner product space and  $a, b \in H$  two linearly independent vectors. If  $x \in H$  is such that

$$\langle x, a \rangle = 0, \text{ and } |\langle x, b \rangle| = 1; \quad (6.2)$$

then one has the inequality

$$\|x\|^2 \geq \frac{\|a\|^2}{\|a\|^2 \|b\|^2 - |\langle a, b \rangle|^2}. \quad (6.3)$$

The equality holds in (6.3) if and only if

$$x = \mu \left( b - \frac{\overline{\langle a, b \rangle}}{\|a\|^2} \cdot a \right)$$

where  $\mu \in \mathbb{K}$  ( $\mathbb{K} = \mathbb{R}, \mathbb{C}$ ) is such that

$$|\mu| = \frac{\|a\|^2}{\|a\|^2 \|b\|^2 - |\langle a, b \rangle|^2}. \quad (6.4)$$

**Proof.** We use Schwarz's inequality in the inner product space  $(H; \langle \cdot, \cdot \rangle)$ , i.e.,

$$\|u\|^2 \|v\|^2 \geq |\langle u, v \rangle|^2; \quad u, v \in H \quad (6.5)$$

with equality iff there exists a scalar  $\alpha \in \mathbb{K}$  such that  $u = \alpha v$ .

If we apply (6.5) for

$$u = z - \frac{\langle z, c \rangle}{\|c\|^2} \cdot c, \quad v = d - \frac{\langle d, c \rangle}{\|c\|^2} \cdot c,$$

where  $c \neq 0$  and  $c, d, z \in H$ , we have

$$\left\| z - \frac{\langle z, c \rangle}{\|c\|^2} \cdot c \right\|^2 \left\| d - \frac{\langle d, c \rangle}{\|c\|^2} \cdot c \right\|^2 \geq \left| \left\langle z - \frac{\langle z, c \rangle}{\|c\|^2} \cdot c, d - \frac{\langle d, c \rangle}{\|c\|^2} \cdot c \right\rangle \right|^2 \quad (6.6)$$

with equality iff there is a scalar  $\beta \in \mathbb{K}$  such that

$$z = \frac{\langle z, c \rangle}{\|c\|^2} \cdot c + \beta \left( d - \frac{\langle d, c \rangle}{\|c\|^2} \cdot c \right). \quad (6.7)$$

Since simple calculations show that

$$\begin{aligned} \left\| z - \frac{\langle z, c \rangle}{\|c\|^2} \cdot c \right\|^2 &= \frac{\|z\|^2 \|c\|^2 - |\langle z, c \rangle|^2}{\|c\|^2}, \\ \left\| d - \frac{\langle d, c \rangle}{\|c\|^2} \cdot c \right\|^2 &= \frac{\|d\|^2 \|c\|^2 - |\langle d, c \rangle|^2}{\|c\|^2}, \end{aligned}$$

and

$$\left\langle z - \frac{\langle z, c \rangle}{\|c\|^2} \cdot c, d - \frac{\langle d, c \rangle}{\|c\|^2} \cdot c \right\rangle = \frac{\langle z, d \rangle \|c\|^2 - \langle z, c \rangle \langle c, d \rangle}{\|c\|^2},$$

then, by (6.6), we deduce

$$\begin{aligned} \left[ \|z\|^2 \|c\|^2 - |\langle z, c \rangle|^2 \right] \left[ \|d\|^2 \|c\|^2 - |\langle d, c \rangle|^2 \right] \\ \geq \left| \langle z, d \rangle \|c\|^2 - \langle z, c \rangle \langle c, d \rangle \right|^2, \end{aligned}$$

with equality if and only if there is a  $\beta \in \mathbb{K}$  such that (6.7) holds.

If  $a, x, b$  satisfy (6.2) then by (??) and (6.7) for the choices  $z = x, c = a$  and  $d = b$  we deduce the inequality (6.3) with equality iff there exists a  $\mu \in \mathbb{K}$  such that

$$x = \mu \left( b - \frac{\overline{\langle a, b \rangle}}{\|a\|^2} \cdot a \right)$$

and, by the second condition in (6.2),

$$\left| \mu \left\langle b - \frac{\overline{\langle a, b \rangle}}{\|a\|^2} \cdot a, b \right\rangle \right| = 1. \quad (6.8)$$

Since (6.8) is clearly equivalent to (6.4), the theorem is completely proved. ■

### 6.1.3 Applications for Sequences and Integrals

The following particular cases hold.

1. If  $\mathbf{a}, \mathbf{b}, \mathbf{x} \in \ell^2(\mathbb{K})$ , where  $\ell^2(\mathbb{K}) := \left\{ \mathbf{x} = (x_i)_{i \in \mathbb{N}}, \sum_{i=1}^{\infty} |x_i|^2 < \infty \right\}$ , with  $\mathbf{a}, \mathbf{b}$  linearly independent and

$$\sum_{i=1}^{\infty} x_i \overline{a_i} = 0 \text{ and } \left| \sum_{i=1}^{\infty} x_i \overline{b_i} \right| = 1,$$

then one has the inequality

$$\sum_{i=1}^{\infty} |x_i|^2 \geq \frac{\sum_{i=1}^{\infty} |a_i|^2}{\sum_{i=1}^{\infty} |a_i|^2 \sum_{i=1}^{\infty} |b_i|^2 - \left| \sum_{i=1}^{\infty} a_i \overline{b_i} \right|^2},$$

with equality iff

$$x_i = \mu \left[ b_i - \frac{\sum_{k=1}^{\infty} \overline{a_k} b_k}{\sum_{k=1}^{\infty} |a_k|^2} \cdot a_i \right], \quad i \in \mathbb{N}$$

and  $\mu \in \mathbb{K}$  with the property

$$|\mu| = \frac{\sum_{i=1}^{\infty} |a_i|^2 \sum_{i=1}^{\infty} |b_i|^2 - \left| \sum_{i=1}^{\infty} a_i \overline{b_i} \right|^2}{\sum_{i=1}^{\infty} |a_i|^2}.$$

2. If  $f, g, h \in L^2(\Omega, m)$ , where  $\Omega$  is a measurable space and  $L^2(\Omega, m) := \left\{ f : \Omega \rightarrow \mathbb{K}, \int_{\Omega} |f(x)|^2 dm(x) < \infty \right\}$ , with  $f, g$  linearly independent and

$$\int_{\Omega} h(x) \overline{f(x)} dm(x) = 0, \quad \left| \int_{\Omega} h(x) \overline{g(x)} dm(x) \right| = 1,$$

then one has the inequality

$$\begin{aligned} & \int_{\Omega} |h(x)|^2 dm(x) \\ & \geq \frac{\int_{\Omega} |f(x)|^2 dm(x)}{\int_{\Omega} |f(x)|^2 dm(x) \int_{\Omega} |g(x)|^2 dm(x) - \left| \int_{\Omega} f(x) \overline{g(x)} dm(x) \right|^2} \end{aligned}$$

with equality iff

$$h(x) = \nu \left[ g(x) - \frac{\int_{\Omega} \overline{f(x)} g(x) dm(x)}{\int_{\Omega} |f(x)|^2 dm(x)} \cdot f(x) \right]$$

for  $m$ -a.e.  $x \in \Omega$ , and  $\nu \in \mathbb{K}$  with

$$|\nu| = \frac{\int_{\Omega} |f(x)|^2 dm(x)}{\int_{\Omega} |f(x)|^2 dm(x) \int_{\Omega} |g(x)|^2 dm(x) - \left| \int_{\Omega} f(x) \overline{g(x)} dm(x) \right|^2}.$$

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## 6.2 Another Ostrowski Type Inequality

### 6.2.1 Introduction

Another result due to Ostrowski which is far less known than the one incorporated in Theorem 271 and obtained in the same work [3, p. 130] (see also [2, p. 94]), is the following one.

**Theorem 274** *Let  $\mathbf{a}$ ,  $\mathbf{b}$  and  $\mathbf{x}$  be  $n$ -tuples of real numbers with  $\mathbf{a} \neq 0$  and*

$$\sum_{i=1}^n a_i x_i = 0 \quad \text{and} \quad \sum_{i=1}^n x_i^2 = 1.$$

*Then*

$$\frac{\sum_{i=1}^n a_i^2 \sum_{i=1}^n b_i^2 - (\sum_{i=1}^n a_i b_i)^2}{\sum_{i=1}^n a_i^2} \geq \left( \sum_{i=1}^n b_i x_i \right)^2. \quad (6.9)$$

*If  $\mathbf{a}$  and  $\mathbf{b}$  are not proportional, then the equality holds in (6.9) iff*

$$x_k = q \cdot \frac{b_k \sum_{i=1}^n a_i^2 - a_k \sum_{i=1}^n a_i b_i}{(\sum_{k=1}^n a_k^2)^{\frac{1}{2}} \left[ \sum_{i=1}^n a_i^2 \sum_{i=1}^n b_i^2 - (\sum_{i=1}^n a_i b_i)^2 \right]^{\frac{1}{2}}}, \quad k \in \{1, \dots, n\},$$

*with  $q \in \{-1, 1, \}$ .*

The case of equality which was neither mentioned in [2] nor in [3] is considered in Remark 276.

In the present section, by the use of an elementary argument based on Schwarz's inequality, a natural generalisation in inner-product spaces of (6.9) is given. The case of equality is analyzed. Applications for sequences and integrals are also provided.

### 6.2.2 The General Result

The following theorem holds [1].

**Theorem 275** *Let  $(H, \langle \cdot, \cdot \rangle)$  be a real or complex inner product space and  $a, b \in H$  two linearly independent vectors. If  $x \in H$  is such that*

$$\langle x, a \rangle = 0 \quad \text{and} \quad \|x\| = 1, \quad (i)$$

*then*

$$\frac{\|a\|^2 \|b\|^2 - |\langle a, b \rangle|^2}{\|a\|^2} \geq |\langle x, b \rangle|^2. \quad (6.10)$$

*The equality holds in (6.10) iff*

$$x = \nu \left( b - \frac{\overline{\langle a, b \rangle}}{\|a\|^2} \cdot a \right),$$



where  $\nu \in \mathbb{K}$  ( $\mathbb{C}, \mathbb{R}$ ) is such that

$$|\nu| = \frac{\|a\|}{\left[ \|a\|^2 \|b\|^2 - |\langle a, b \rangle|^2 \right]^{\frac{1}{2}}}.$$

**Proof.** We use Schwarz's inequality in the inner product space  $H$ , i.e.,

$$\|u\|^2 \|v\|^2 \geq |\langle u, v \rangle|^2, \quad u, v \in H \quad (6.11)$$

with equality iff there is a scalar  $\alpha \in \mathbb{K}$  such that

$$u = \alpha v.$$

If we apply (6.11) for  $u = z - \frac{\langle z, c \rangle}{\|c\|^2} \cdot c$ ,  $v = d - \frac{\langle d, c \rangle}{\|c\|^2} \cdot c$ , where  $c \neq 0$  and  $c, d, z \in H$ , then we deduce the inequality

$$\begin{aligned} \left[ \|z\|^2 \|c\|^2 - |\langle z, c \rangle|^2 \right] \left[ \|d\|^2 \|c\|^2 - |\langle d, c \rangle|^2 \right] \\ \geq \left| \langle z, d \rangle \|c\|^2 - \langle z, c \rangle \langle c, d \rangle \right|^2 \end{aligned} \quad (6.12)$$

with equality iff there is a  $\beta \in \mathbb{K}$  such that

$$z = \frac{\langle z, c \rangle}{\|c\|^2} \cdot c + \beta \left( d - \frac{\langle d, c \rangle}{\|c\|^2} \cdot c \right).$$

If in (6.12) we choose  $z = x$ ,  $c = a$  and  $d = b$ , where  $a$  and  $x$  satisfy (i), then we deduce

$$\|a\|^2 \left[ \|a\|^2 \|b\|^2 - |\langle a, b \rangle|^2 \right] \geq \left[ \langle x, b \rangle \|a\|^2 \right]^2$$

which is clearly equivalent to (6.10).

The equality holds in (6.10) iff

$$x = \nu \left( b - \frac{\overline{\langle a, b \rangle}}{\|a\|^2} \cdot a \right),$$

where  $\nu \in \mathbb{K}$  satisfies the condition

$$1 = \|x\| = |\nu| \left\| b - \frac{\overline{\langle a, b \rangle}}{\|a\|^2} \cdot a \right\| = |\nu| \left[ \frac{\|a\|^2 \|b\|^2 - |\langle a, b \rangle|^2}{\|a\|^2} \right]^{\frac{1}{2}},$$

and the theorem is thus proved. ■

### 6.2.3 Applications for Sequences and Integrals

The following particular cases hold.

1. If  $\mathbf{a}, \mathbf{b}, \mathbf{x} \in \ell^2(\mathbb{K})$ ,  $\mathbb{K} = \mathbb{C}, \mathbb{R}$ , where

$$\ell^2(\mathbb{K}) := \left\{ \mathbf{x} = (x_i)_{i \in \mathbb{N}}, \sum_{i=1}^{\infty} |x_i|^2 < \infty \right\}$$

with  $\mathbf{a}, \mathbf{b}$  linearly independent and

$$\sum_{i=1}^{\infty} x_i \overline{a_i} = 0, \quad \sum_{i=1}^{\infty} |x_i|^2 = 1,$$

then

$$\frac{\sum_{i=1}^{\infty} |a_i|^2 \sum_{i=1}^{\infty} |b_i|^2 - \left| \sum_{i=1}^{\infty} a_i \overline{b_i} \right|^2}{\sum_{i=1}^{\infty} |a_i|^2} \geq \left| \sum_{i=1}^{\infty} x_i \overline{b_i} \right|^2. \quad (6.13)$$

The equality holds in (6.13) iff

$$x_i = \nu \left[ b_i - \frac{\sum_{k=1}^{\infty} a_k \overline{b_k}}{\sum_{k=1}^{\infty} |a_k|^2} \cdot a_i \right], \quad i \in \{1, 2, \dots\}$$

with  $\nu \in \mathbb{K}$  such that

$$|\nu| = \frac{\left( \sum_{k=1}^{\infty} |a_k|^2 \right)^{\frac{1}{2}}}{\left[ \sum_{k=1}^{\infty} |a_k|^2 \sum_{k=1}^{\infty} |b_k|^2 - \left| \sum_{k=1}^{\infty} a_k \overline{b_k} \right|^2 \right]^{\frac{1}{2}}}.$$

**Remark 276** The case of equality in (6.9) is obviously a particular case of the above. We omit the details.

2. If  $f, g, h \in L^2(\Omega, m)$ , where  $\Omega$  is an  $m$ -measurable space and

$$L^2(\Omega, m) := \left\{ f : \Omega \rightarrow \mathbb{K}, \int_{\Omega} |f(x)|^2 dm(x) < \infty \right\},$$

with  $f, g$  being linearly independent and

$$\int_{\Omega} h(x) \overline{f(x)} dm(x) = 0, \quad \int_{\Omega} |h(x)|^2 dm(x) = 1,$$

then

$$\begin{aligned} & \frac{\int_{\Omega} |f(x)|^2 dm(x) \int_{\Omega} |g(x)|^2 dm(x) - \left| \int_{\Omega} f(x) \overline{g(x)} dm(x) \right|^2}{\int_{\Omega} |f(x)|^2 dm(x)} \\ & \geq \left| \int_{\Omega} h(x) \overline{g(x)} dm(x) \right|^2. \end{aligned} \quad (6.14)$$

The equality holds in (6.14) iff

$$h(x) = \nu \left[ g(x) - \frac{\int_{\Omega} g(x) \overline{f(x)} dm(x)}{\int_{\Omega} |f(x)|^2 dm(x)} f(x) \right] \quad \text{for a.e. } x \in \Omega$$

and  $\nu \in \mathbb{K}$  with

$$|\nu| = \frac{\left( \int_{\Omega} |f(x)|^2 dm(x) \right)^{\frac{1}{2}}}{\left[ \int_{\Omega} |f(x)|^2 dm(x) \int_{\Omega} |g(x)|^2 dm(x) - \left| \int_{\Omega} f(x) \overline{g(x)} dm(x) \right|^2 \right]^{\frac{1}{2}}}.$$

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## 6.3 The Wagner Inequality in Inner Product Spaces

### 6.3.1 Introduction

In 1965, S.S. Wagner [4] (see also [2] or [3, p. 85]) pointed out the following generalisation of Cauchy-Bunyakovsky-Schwarz's inequality for real numbers.

**Theorem 277** *Let  $\mathbf{a} = (a_1, \dots, a_n)$  and  $\mathbf{b} = (b_1, \dots, b_n)$  be two  $n$ -tuples of real numbers. Then for any  $x \in [0, 1]$ , one has the inequality*

$$\begin{aligned} & \left( \sum_{k=1}^n a_k b_k + x \cdot \sum_{1 \leq i \neq j \leq n} a_i b_j \right)^2 \\ & \leq \left( \sum_{k=1}^n a_k^2 + 2x \cdot \sum_{1 \leq i < j \leq n} a_i a_j \right) \cdot \left( \sum_{k=1}^n b_k^2 + 2x \cdot \sum_{1 \leq i < j \leq n} b_i b_j \right). \end{aligned}$$

For  $x = 0$ , we recapture the Cauchy-Bunyakovsky-Schwarz's inequality, i.e., (see for example [3, p. 84])

$$\left( \sum_{k=1}^n a_k b_k \right)^2 \leq \sum_{k=1}^n a_k^2 \sum_{k=1}^n b_k^2,$$

with equality if and only if there exists a real number  $r$  such that  $a_k = r b_k$  for each  $k \in \{1, \dots, n\}$ .

In this section we extend the above result for sequences of vectors in real or complex inner product spaces.

### 6.3.2 The Results

Let  $(H; \langle \cdot, \cdot \rangle)$  be an inner product space over  $\mathbb{K}$ , where  $\mathbb{K} = \mathbb{R}$  or  $\mathbb{K} = \mathbb{C}$ . The following result holds [1].

**Theorem 278** *Let  $\mathbf{x} = (x_1, \dots, x_n)$  and  $\mathbf{y} = (y_1, \dots, y_n)$  be two  $n$ -tuples of vectors in  $H$ . Then for any  $\alpha \in [0, 1]$  one has the inequality*

$$\begin{aligned} & \left[ \sum_{k=1}^n \operatorname{Re} \langle x_k, y_k \rangle + \alpha \cdot \sum_{1 \leq i \neq j \leq n} \operatorname{Re} \langle x_i, y_j \rangle \right]^2 \tag{6.15} \\ & \leq \left[ \sum_{k=1}^n \|x_k\|^2 + 2\alpha \cdot \sum_{1 \leq i < j \leq n} \operatorname{Re} \langle x_i, x_j \rangle \right] \left[ \sum_{k=1}^n \|y_k\|^2 + 2\alpha \cdot \sum_{1 \leq i < j \leq n} \operatorname{Re} \langle y_i, y_j \rangle \right]. \end{aligned}$$

**Proof.** Following the proof by P. Flor [2], we may consider the function  $f : \mathbb{R} \rightarrow \mathbb{R}$ , given by

$$f(t) = (1 - \alpha) \cdot \sum_{k=1}^n \|tx_k - y_k\|^2 + \alpha \cdot \left\| \sum_{k=1}^n (tx_k - y_k) \right\|^2. \tag{6.16}$$

Then

$$\begin{aligned}
f(t) = & \left[ (1-\alpha) \cdot \sum_{k=1}^n \|x_k\|^2 + \alpha \cdot \left\| \sum_{k=1}^n x_k \right\|^2 \right] t^2 \\
& + 2 \left[ (1-\alpha) \cdot \sum_{k=1}^n \operatorname{Re} \langle x_k, y_k \rangle + \alpha \cdot \operatorname{Re} \left\langle \sum_{k=1}^n x_k, \sum_{k=1}^n y_k \right\rangle \right] t \\
& + \left[ (1-\alpha) \cdot \sum_{k=1}^n \|y_k\|^2 + \alpha \cdot \left\| \sum_{k=1}^n y_k \right\|^2 \right]. \quad (6.17)
\end{aligned}$$

Observe that

$$\left\| \sum_{k=1}^n x_k \right\|^2 = \sum_{k=1}^n \|x_k\|^2 + 2 \cdot \sum_{1 \leq i < j \leq n} \operatorname{Re} \langle x_i, x_j \rangle \quad (6.18)$$

and

$$\left\| \sum_{k=1}^n y_k \right\|^2 = \sum_{k=1}^n \|y_k\|^2 + 2 \cdot \sum_{1 \leq i < j \leq n} \operatorname{Re} \langle y_i, y_j \rangle. \quad (6.19)$$

Also

$$\operatorname{Re} \left\langle \sum_{k=1}^n x_k, \sum_{k=1}^n y_k \right\rangle = \sum_{k=1}^n \operatorname{Re} \langle x_k, y_k \rangle + \sum_{1 \leq i \neq j \leq n} \operatorname{Re} \langle x_i, y_j \rangle. \quad (6.20)$$

Using (6.17) – (6.20), we deduce that

$$\begin{aligned}
f(t) = & \left[ \sum_{k=1}^n \|x_k\|^2 + 2\alpha \cdot \sum_{1 \leq i < j \leq n} \operatorname{Re} \langle x_i, x_j \rangle \right] t^2 \\
& + 2 \left[ \sum_{k=1}^n \operatorname{Re} \langle x_k, y_k \rangle + \alpha \cdot \sum_{1 \leq i \neq j \leq n} \operatorname{Re} \langle x_i, y_j \rangle \right] t \\
& + \left[ \sum_{k=1}^n \|y_k\|^2 + 2\alpha \cdot \sum_{1 \leq i < j \leq n} \operatorname{Re} \langle y_i, y_j \rangle \right]. \quad (6.21)
\end{aligned}$$

Since, by (6.16),  $f(t) \geq 0$  for any  $t \in \mathbb{R}$ , it follows that the discriminant of the quadratic function given by (6.21) is negative, which is clearly equivalent with the desired inequality (6.15). ■

One may obtain an interesting inequality if  $\mathbf{x}$  and  $\mathbf{y}$  are assumed to incorporate orthogonal vectors.

**Corollary 279** Assume that  $\{x_i\}_{i=1,\dots,n}$  are orthogonal, i.e.,  $x_i \perp x_j$  for any  $i, j \in \{1, \dots, n\}$ ,  $i \neq j$ ; and  $\{y_i\}_{i=1,\dots,n}$  are also orthogonal in the real inner product space  $(H; \langle \cdot, \cdot \rangle)$ . Then

$$\sup_{\alpha \in [0,1]} \left[ \sum_{k=1}^n \langle x_k, y_k \rangle + \alpha \cdot \sum_{1 \leq i \neq j \leq n} \langle x_i, y_j \rangle \right]^2 \leq \sum_{k=1}^n \|x_k\|^2 \sum_{k=1}^n \|y_k\|^2.$$

### 6.3.3 Applications

1. If we assume that  $H = \mathbb{C}$ , with the inner product  $\langle x, y \rangle = x \cdot \bar{y}$ , then by (6.15) we may deduce the following Wagner type inequality for complex numbers

$$\begin{aligned} & \left[ \sum_{k=1}^n \operatorname{Re}(a_k \bar{b}_k) + \alpha \cdot \sum_{1 \leq i \neq j \leq n} \operatorname{Re}(a_i \bar{b}_j) \right]^2 \\ & \leq \left[ \sum_{k=1}^n |a_k|^2 + 2\alpha \cdot \sum_{1 \leq i < j \leq n} \operatorname{Re}(a_i \bar{a}_j) \right] \\ & \quad \times \left[ \sum_{k=1}^n |b_k|^2 + 2\alpha \cdot \sum_{1 \leq i < j \leq n} \operatorname{Re}(b_i \bar{b}_j) \right], \end{aligned}$$

where  $\alpha \in [0, 1]$  and  $\mathbf{a} = (a_1, \dots, a_n)$ ,  $\mathbf{b} = (b_1, \dots, b_n) \in \mathbb{C}^n$ .

2. Consider the Hilbert space  $L_2(\Omega, \mu) := \left\{ f : \Omega \rightarrow \mathbb{C}, \int_{\Omega} |f(x)|^2 d\mu(x) < \infty \right\}$ , where  $\Omega$  is a  $\mu$ -measurable space and  $\mu : \Omega \rightarrow [0, \infty]$  is a positive measure on  $\Omega$ . Then for  $H = L_2(\Omega, \mu)$  and since the inner product generating the norm is given by

$$\langle f, g \rangle = \int_{\Omega} f(x) \bar{g}(x) d\mu(x),$$

we get the inequality

$$\begin{aligned} & \left[ \sum_{k=1}^n \int_{\Omega} \operatorname{Re}(f_k(x) \bar{g}_k(x)) d\mu(x) + \alpha \cdot \sum_{1 \leq i \neq j \leq n} \int_{\Omega} \operatorname{Re}(f_i(x) \bar{g}_j(x)) d\mu(x) \right]^2 \\ & \leq \left[ \sum_{k=1}^n \int_{\Omega} |f_k(x)|^2 d\mu(x) + 2\alpha \cdot \sum_{1 \leq i < j \leq n} \int_{\Omega} \operatorname{Re}(f_i(x) \bar{f}_j(x)) d\mu(x) \right] \\ & \quad \times \left[ \sum_{k=1}^n \int_{\Omega} |g_k(x)|^2 d\mu(x) + 2\alpha \cdot \sum_{1 \leq i < j \leq n} \int_{\Omega} \operatorname{Re}(g_i(x) \bar{g}_j(x)) d\mu(x) \right], \end{aligned}$$

where  $f_i, g_i \in L_2(\Omega, \mu)$ ,  $i \in \{1, \dots, n\}$  and  $\alpha \in [0, 1]$ .

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## 6.4 A Monotonicity Property of Bessel's Inequality

Let  $X$  be a linear space over the real or complex number field  $\mathbb{K}$ . A mapping  $(\cdot, \cdot) : X \times X \rightarrow \mathbb{K}$  is said to be of *positive hermitian form* if the following conditions are satisfied:

- (i)  $(\alpha x + \beta y, z) = \alpha (x, z) + \beta (y, z)$  for all  $x, y, z \in X$  and  $\alpha, \beta \in \mathbb{K}$ ;
- (ii)  $(y, x) = \overline{(x, y)}$  for all  $x, y \in X$ ;
- (iii)  $(x, x) \geq 0$  for all  $x \in X$ .

If  $\|x\| := (x, x)^{\frac{1}{2}}$  denotes the semi-norm associated to this form and  $(e_i)_{i \in I}$  is an orthonormal family of vectors in  $X$ , that is,  $(e_i, e_j) = \delta_{ij}$  ( $i, j \in I$ ), then one has [4]:

$$\|x\|^2 \geq \sum_{i \in I} |(x, e_i)|^2 \quad \text{for all } x \in X, \quad (6.22)$$

which is well known in the literature as Bessel's inequality.

The main aim of the section is to point out an improvement for this result as follows [1].

**Theorem 280** *Let  $X$  be a linear space and  $(\cdot, \cdot)_2, (\cdot, \cdot)_1$  two hermitian forms on  $X$  such that  $\|\cdot\|_2$  is greater than or equal to  $\|\cdot\|_1$ , that is,  $\|x\|_2 \geq \|x\|_1$  for all  $x \in X$ . Assume that  $(e_i)_{i \in I}$  is an orthonormal family in  $(X; (\cdot, \cdot)_2)$  and  $(f_j)_{j \in J}$  is an orthonormal family in  $(X; (\cdot, \cdot)_1)$  such that for any  $i \in I$  there exists a finite  $K \subset J$  such that*

$$e_i = \sum_{j \in K} \alpha_j f_j, \quad \alpha_j \in \mathbb{K}, \quad (j \in K), \quad (6.23)$$

*then one has the inequality:*

$$\|x\|_2^2 - \sum_{i \in I} |(x, e_i)_2|^2 \geq \|x\|_1^2 - \sum_{j \in J} |(x, f_j)_1|^2 \geq 0, \quad \text{for all } x \in X. \quad (6.24)$$

In order to prove this, we require the following lemma.

**Lemma 281** *Let  $X$  be a linear space endowed with a positive hermitian form  $(\cdot, \cdot)$  and  $(g_k)$ ,  $k \in \{1, \dots, n\}$  be an orthonormal family in  $(X; (\cdot, \cdot))$ . Then*

$$\left\| x - \sum_{k=1}^n \lambda_k g_k \right\|^2 \geq \|x\|^2 - \sum_{k=1}^n |(x, g_k)|^2 \geq 0,$$

*for all  $\lambda_k \in \mathbb{K}$ ,  $k \in \{1, \dots, n\}$  and  $x \in X$ .*

The proof follows by mathematical induction.

**Proof of Theorem 280.** Let  $H$  be a finite subset of  $I$ . Since  $\|\cdot\|_2$  is greater than  $\|\cdot\|_1$ , we have:

$$\|x\|_2^2 - \sum_{i \in H} |(x, e_i)_2|^2 = \left\| x - \sum_{i \in H} (x, e_i)_2 e_i \right\|_2^2 \geq \left\| x - \sum_{i \in H} (x, e_i)_2 e_i \right\|_1^2, \quad x \in X.$$

Since, by (6.23), we may state that for any  $i \in H$  there exists a finite  $K \subset J$  with

$$e_i = \sum_{j \in K} (e_i, f_j)_1 f_j,$$

we have, for all  $x \in X$

$$\begin{aligned} \left\| x - \sum_{i \in H} (x, e_i)_2 e_i \right\|_1^2 &= \left\| x - \sum_{i \in H} (x, e_i)_2 \sum_{j \in K} (e_i, f_j)_1 f_j \right\|_1^2 \\ &= \left\| x - \sum_{j \in K} \left( \sum_{i \in H} (x, e_i)_2 (e_i, f_j)_1 \right) f_j \right\|_1^2. \end{aligned}$$

Applying Lemma 281, we can conclude that

$$\left\| x - \sum_{j \in K} \lambda_j f_j \right\|_1^2 \geq \|x\|_1^2 - \sum_{j \in K} |(x, f_j)_1|^2, \quad x \in X,$$

where

$$\lambda_j = \left( \sum_{i \in H} (x, e_i)_2 (e_i, f_j)_1 \right)_1 \in \mathbb{K}, \quad (j \in K).$$

Consequently, we have

$$\|x\|_2^2 - \sum_{i \in H} |(x, e_i)_2|^2 \geq \|x\|_1^2 - \sum_{j \in K} |(x, f_j)_1|^2 \geq \|x\|_1^2 - \sum_{j \in J} |(x, f_j)_1|^2$$

for all  $x \in X$  and  $H$  a finite subset of  $I$ , from which (6.24) results. ■

**Corollary 282** Let  $\|\cdot\|_1, \|\cdot\|_2 : X \rightarrow \mathbb{R}_+$  be as above. Then for all  $x, y \in X$ , we have the inequality:

$$\|x\|_2^2 \|y\|_2^2 - |(x, y)_2|^2 \geq \|x\|_1^2 \|y\|_1^2 - |(x, y)_1|^2 \geq 0, \quad (6.25)$$

which is an improvement of the well known Cauchy-Schwarz inequality.

**Remark 283** For a different proof of (6.25), see also [2] or [3].

Now, we will give some natural applications of the above theorem.

1. Let  $(X; (\cdot, \cdot))$  be an inner product space and  $(e_i)_{i \in I}$  an orthonormal family in  $X$ . Assume that  $A : X \rightarrow X$  is a linear operator such that  $\|Ax\| \leq \|x\|$  for all  $x \in X$  and  $(Ae_i, Ae_j) = \delta_{ij}$  for all  $i, j \in I$ . Then one has the inequality

$$\|x\|^2 - \sum_{i \in I} |(x, e_i)|^2 \geq \|Ax\|^2 - \sum_{i \in I} |(Ax, Ae_i)|^2 \geq 0, \text{ for all } x \in X.$$

2. If  $A : X \rightarrow X$  is such that  $\|Ax\| \geq \|x\|$  for all  $x \in X$ , then, with the previous assumptions, we also have

$$0 \leq \|x\|^2 - \sum_{i \in I} |(x, e_i)|^2 \leq \|Ax\|^2 - \sum_{i \in I} |(Ax, Ae_i)|^2, \text{ for all } x \in X.$$

3. Suppose that  $A : X \rightarrow X$  is a symmetric positive definite operator with  $(Ax, x) \geq \|x\|^2$  for all  $x \in X$ . If  $(e_i)_{i \in I}$  is an orthonormal family in  $X$  such that  $(Ae_i, Ae_j) = \delta_{ij}$  for all  $i, j \in I$ , then one has the inequality

$$0 \leq \|x\|^2 - \sum_{i \in I} |(x, e_i)|^2 \leq (Ax, x) - \sum_{i \in I} |(Ax, e_i)|^2,$$

for any  $x \in X$ .

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## 6.5 Other Bombieri Type Inequalities

### 6.5.1 Introduction

In 1971, E. Bombieri [1] gave the following generalisation of Bessel's inequality:

$$\sum_{i=1}^n |(x, y_i)|^2 \leq \|x\|^2 \max_{1 \leq i \leq n} \left\{ \sum_{j=1}^n |(y_i, y_j)| \right\}, \quad (6.26)$$

where  $x, y_1, \dots, y_n$  are vectors in the inner product space  $(H; (\cdot, \cdot))$ .

It is obvious that if  $(y_i)_{1 \leq i \leq n} = (e_i)_{1 \leq i \leq n}$ , where  $(e_i)_{1 \leq i \leq n}$  are orthonormal vectors in  $H$ , i.e.,  $(e_i, e_j) = \delta_{ij}$  ( $i, j = 1, \dots, n$ ), where  $\delta_{ij}$  is the Kronecker delta, then (6.26) provides Bessel's inequality

$$\sum_{i=1}^n |(x, e_i)|^2 \leq \|x\|^2, \quad x \in H.$$

In this section we point out some Bombieri type inequalities that complement the results obtained in Chapter 4.

### 6.5.2 The Results

The following lemma, which is of interest in itself, holds [2].

**Lemma 284** *Let  $z_1, \dots, z_n \in H$  and  $\alpha_1, \dots, \alpha_n \in \mathbb{K}$ . Then one has the inequalities:*

$$\begin{aligned} & \left\| \sum_{i=1}^n \alpha_i z_i \right\|^2 \\ & \leq \begin{cases} \max_{1 \leq i \leq n} |\alpha_i|^2 \sum_{i,j=1}^n |(z_i, z_j)|; \\ (\sum_{i=1}^n |\alpha_i|^p)^{\frac{2}{p}} \left( \sum_{i,j=1}^n |(z_i, z_j)|^q \right)^{\frac{1}{q}}, \quad \text{where } p > 1, \frac{1}{p} + \frac{1}{q} = 1; \\ (\sum_{i=1}^n |\alpha_i|)^2 \max_{1 \leq i,j \leq n} |(z_i, z_j)|; \end{cases} \\ & \leq \begin{cases} \max_{1 \leq i \leq n} |\alpha_i|^2 (\sum_{i=1}^n \|z_i\|)^2; \\ (\sum_{i=1}^n |\alpha_i|^p)^{\frac{2}{p}} (\sum_{i=1}^n \|z_i\|^q)^{\frac{2}{q}}, \quad \text{where } p > 1, \frac{1}{p} + \frac{1}{q} = 1; \\ (\sum_{i=1}^n |\alpha_i|)^2 \max_{1 \leq i \leq n} \|z_i\|^2. \end{cases} \end{aligned}$$

**Proof.** We observe that

$$\begin{aligned} \left\| \sum_{i=1}^n \alpha_i z_i \right\|^2 &= \left( \sum_{i=1}^n \alpha_i z_i, \sum_{j=1}^n \alpha_j z_j \right) = \sum_{i=1}^n \sum_{j=1}^n \alpha_i \overline{\alpha_j} (z_i, z_j) \\ &= \left| \sum_{i=1}^n \sum_{j=1}^n \alpha_i \overline{\alpha_j} (z_i, z_j) \right| \leq \sum_{i=1}^n \sum_{j=1}^n |\alpha_i| |\alpha_j| |(z_i, z_j)| =: M. \end{aligned}$$

Firstly, we have

$$\begin{aligned} M &\leq \max_{1 \leq i, j \leq n} \{|\alpha_i| |\alpha_j|\} \sum_{i, j=1}^n |(z_i, z_j)| \\ &= \max_{1 \leq i \leq n} |\alpha_i|^2 \sum_{i, j=1}^n |(z_i, z_j)|. \end{aligned}$$

Secondly, by the Hölder inequality for double sums, we obtain

$$\begin{aligned} M &\leq \left[ \sum_{i, j=1}^n (|\alpha_i| |\alpha_j|)^p \right]^{\frac{1}{p}} \left( \sum_{i, j=1}^n |(z_i, z_j)|^q \right)^{\frac{1}{q}} \\ &= \left( \sum_{i=1}^n |\alpha_i|^p \sum_{j=1}^n |\alpha_j|^p \right)^{\frac{1}{p}} \left( \sum_{i, j=1}^n |(z_i, z_j)|^q \right)^{\frac{1}{q}} \\ &= \left( \sum_{i=1}^n |\alpha_i|^p \right)^{\frac{2}{p}} \left( \sum_{i, j=1}^n |(z_i, z_j)|^q \right)^{\frac{1}{q}}, \end{aligned}$$

where  $p > 1$ ,  $\frac{1}{p} + \frac{1}{q} = 1$ .

Finally, we have

$$M \leq \max_{1 \leq i, j \leq n} |(z_i, z_j)| \sum_{i, j=1}^n |\alpha_i| |\alpha_j| = \left( \sum_{i=1}^n |\alpha_i| \right)^2 \max_{1 \leq i, j \leq n} |(z_i, z_j)|$$

and the first part of the lemma is proved.

The second part is obvious on taking into account, by Schwarz's inequality in  $H$ , that we have

$$|(z_i, z_j)| \leq \|z_i\| \|z_j\|,$$

for any  $i, j \in \{1, \dots, n\}$ . We omit the details. ■

**Corollary 285** *With the assumptions in Lemma 284, one has*

$$\begin{aligned} \left\| \sum_{i=1}^n \alpha_i z_i \right\|^2 &\leq \sum_{i=1}^n |\alpha_i|^2 \left( \sum_{i,j=1}^n |(z_i, z_j)|^2 \right)^{\frac{1}{2}} \\ &\leq \sum_{i=1}^n |\alpha_i|^2 \sum_{i=1}^n \|z_i\|^2. \end{aligned} \quad (6.27)$$

The proof follows by Lemma 284 on choosing  $p = q = 2$ .

Note also that (6.27) provides a refinement of the well known Cauchy-Bunyakovsky-Schwarz inequality for sequences of vectors in inner product spaces, namely

$$\left\| \sum_{i=1}^n \alpha_i z_i \right\|^2 \leq \sum_{i=1}^n |\alpha_i|^2 \sum_{i=1}^n \|z_i\|^2.$$

The following lemma also holds [2].

**Lemma 286** *Let  $x, y_1, \dots, y_n \in H$  and  $c_1, \dots, c_n \in \mathbb{K}$ . Then one has the inequalities:*

$$\begin{aligned} &\left| \sum_{i=1}^n c_i (x, y_i) \right|^2 \\ &\leq \|x\|^2 \times \begin{cases} \max_{1 \leq i \leq n} |c_i|^2 \sum_{i,j=1}^n |(y_i, y_j)|; \\ (\sum_{i=1}^n |c_i|^p)^{\frac{2}{p}} \left( \sum_{i,j=1}^n |(y_i, y_j)|^q \right)^{\frac{1}{q}}, \\ \quad \text{where } p > 1, \frac{1}{p} + \frac{1}{q} = 1; \\ (\sum_{i=1}^n |c_i|)^2 \max_{1 \leq i,j \leq n} |(y_i, y_j)|; \end{cases} \\ &\leq \|x\|^2 \times \begin{cases} \max_{1 \leq i \leq n} |c_i|^2 (\sum_{i=1}^n \|y_i\|)^2; \\ (\sum_{i=1}^n |c_i|^p)^{\frac{2}{p}} (\sum_{i=1}^n \|y_i\|^q)^{\frac{2}{q}}, \\ \quad \text{where } p > 1, \frac{1}{p} + \frac{1}{q} = 1; \\ (\sum_{i=1}^n |c_i|)^2 \max_{1 \leq i \leq n} \|y_i\|^2. \end{cases} \end{aligned} \quad (6.28)$$

**Proof.** We have, by Schwarz's inequality in the inner product  $(H; (\cdot, \cdot))$ , that

$$\left| \sum_{i=1}^n c_i (x, y_i) \right|^2 = \left| \left( x, \sum_{i=1}^n \overline{c_i} y_i \right) \right|^2 \leq \|x\|^2 \left\| \sum_{i=1}^n \overline{c_i} y_i \right\|^2.$$

Now, applying Lemma 284 for  $\alpha_i = \overline{c_i}$ ,  $z_i = y_i$  ( $i = 1, \dots, n$ ), the inequality (6.28) is proved. ■

**Corollary 287** *With the assumptions in Lemma 286, one has*

$$\begin{aligned} \left| \sum_{i=1}^n c_i(x, y_i) \right|^2 &\leq \|x\|^2 \sum_{i=1}^n |c_i|^2 \left( \sum_{i,j=1}^n |(y_i, y_j)|^2 \right)^{\frac{1}{2}} \\ &\leq \|x\|^2 \sum_{i=1}^n |c_i|^2 \sum_{i=1}^n \|y_i\|^2. \end{aligned} \quad (6.29)$$

The proof follows by Lemma 286, on choosing  $p = q = 2$ .

**Remark 288** *The inequality (6.29) was firstly obtained in [3, Inequality (7)].*

The following theorem incorporating three Bombieri type inequalities holds [2].

**Theorem 289** *Let  $x, y_1, \dots, y_n \in H$ . Then one has the inequalities:*

$$\begin{aligned} &\sum_{i=1}^n |(x, y_i)|^2 \\ &\leq \|x\| \times \begin{cases} \max_{1 \leq i \leq n} |(x, y_i)| \left( \sum_{i,j=1}^n |(y_i, y_j)| \right)^{\frac{1}{2}}; \\ \left( \sum_{i=1}^n |(x, y_i)|^p \right)^{\frac{1}{p}} \left( \sum_{i,j=1}^n |(y_i, y_j)|^q \right)^{\frac{1}{2q}}, \\ \quad \text{where } p > 1, \frac{1}{p} + \frac{1}{q} = 1; \\ \sum_{i=1}^n |(x, y_i)| \max_{1 \leq i,j \leq n} |(y_i, y_j)|^{\frac{1}{2}}. \end{cases} \end{aligned} \quad (6.30)$$

**Proof.** Choosing  $c_i = \overline{(x, y_i)}$  ( $i = 1, \dots, n$ ) in (6.28) we deduce

$$\begin{aligned} &\left( \sum_{i=1}^n |(x, y_i)|^2 \right)^2 \\ &\leq \|x\|^2 \times \begin{cases} \max_{1 \leq i \leq n} |(x, y_i)|^2 \left( \sum_{i,j=1}^n |(y_i, y_j)| \right); \\ \left( \sum_{i=1}^n |(x, y_i)|^p \right)^{\frac{2}{p}} \left( \sum_{i,j=1}^n |(y_i, y_j)|^q \right)^{\frac{1}{q}}, \\ \quad \text{where } p > 1, \frac{1}{p} + \frac{1}{q} = 1; \\ \left( \sum_{i=1}^n |(x, y_i)| \right)^2 \max_{1 \leq i,j \leq n} |(y_i, y_j)|; \end{cases} \end{aligned} \quad (6.31)$$

which, by taking the square root, is clearly equivalent to (6.30). ■



**Remark 290** If  $(y_i)_{1 \leq i \leq n} = (e_i)_{1 \leq i \leq n}$ , where  $(e_i)_{1 \leq i \leq n}$  are orthonormal vectors in  $H$ , then by (6.30) we deduce

$$\sum_{i=1}^n |(x, e_i)|^2 \leq \|x\| \begin{cases} \sqrt{n} \max_{1 \leq i \leq n} |(x, e_i)|; \\ n^{\frac{1}{2q}} \left( \sum_{i=1}^n |(x, e_i)|^p \right)^{\frac{1}{p}}, \\ \text{where } p > 1, \frac{1}{p} + \frac{1}{q} = 1; \\ \sum_{i=1}^n |(x, e_i)|. \end{cases} \quad (6.32)$$

If in (6.31) we take  $p = q = 2$ , then we obtain the following inequality which was formulated in [3, p. 81].

**Corollary 291** With the assumptions in Theorem 289, we have:

$$\sum_{i=1}^n |(x, y_i)|^2 \leq \|x\|^2 \left( \sum_{i,j=1}^n |(y_i, y_j)|^2 \right)^{\frac{1}{2}}. \quad (6.33)$$

**Remark 292** Observe, that by the monotonicity of power means, we may write

$$\left( \frac{\sum_{i=1}^n |(x, y_i)|^p}{n} \right)^{\frac{1}{p}} \leq \left( \frac{\sum_{i=1}^n |(x, y_i)|^2}{n} \right)^{\frac{1}{2}}, \quad 1 < p \leq 2.$$

Taking the square in both sides, one has

$$\left( \frac{\sum_{i=1}^n |(x, y_i)|^p}{n} \right)^{\frac{2}{p}} \leq \frac{\sum_{i=1}^n |(x, y_i)|^2}{n},$$

giving

$$\left( \sum_{i=1}^n |(x, y_i)|^p \right)^{\frac{2}{p}} \leq n^{\frac{2}{p}-1} \sum_{i=1}^n |(x, y_i)|^2. \quad (6.34)$$

Using (6.34) and the second inequality in (6.31) we may deduce the following result

$$\sum_{i=1}^n |(x, y_i)|^2 \leq n^{\frac{2}{p}-1} \|x\|^2 \left( \sum_{i,j=1}^n |(y_i, y_j)|^q \right)^{\frac{1}{q}}, \quad (6.35)$$

for  $1 < p \leq 2, \frac{1}{p} + \frac{1}{q} = 1$ .

Note that for  $p = 2$  ( $q = 2$ ) we recapture (6.33).

**Remark 293** Let us compare Bombieri's result

$$\sum_{i=1}^n |(x, y_i)|^2 \leq \|x\|^2 \max_{1 \leq i \leq n} \left\{ \sum_{j=1}^n |(y_i, y_j)| \right\} \quad (6.36)$$

with our general result (6.35).

To do that, denote

$$M_1 := \max_{1 \leq i \leq n} \left\{ \sum_{j=1}^n |(y_i, y_j)| \right\}$$

and

$$M_2 := n^{\frac{2}{p}-1} \left( \sum_{i,j=1}^n |(y_i, y_j)|^q \right)^{\frac{1}{q}}, \quad 1 < p \leq 2, \quad \frac{1}{p} + \frac{1}{q} = 1.$$

Consider the inner product space  $H = \mathbb{R}$ ,  $(x, y) = x \cdot y$ ,  $n = 2$  and  $y_1 = a > 0$ ,  $y_2 = b > 0$ . Then

$$M_1 = \max \{a^2 + ab, ab + b^2\} = (a + b) \max \{a, b\},$$

$$M_2 = 2^{\frac{2}{p}-1} (a^q + b^q)^{\frac{2}{q}} = 2^{\frac{2}{p}-1} \left( a^{\frac{p}{p-1}} + b^{\frac{p}{p-1}} \right)^{\frac{2(p-1)}{p}}, \quad 1 < p \leq 2.$$

Assume that  $a = 1$ ,  $b \in [0, 1]$ ,  $p \in (1, 2]$ . Utilizing Maple 6, one may easily see by plotting the function

$$f(b, p) := M_2 - M_1 = 2^{\frac{2}{p}-1} \left( 1 + b^{\frac{p}{p-1}} \right)^{\frac{2(p-1)}{p}} - 1 - b$$

that it has positive and negative values in the box  $[0, 1] \times [1, 2]$ , showing that the inequalities (6.35) and (6.36) cannot be compared. This means that one is not always better than the other.

# Bibliography

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## 6.6 Some Pre-Grüss Inequalities

### 6.6.1 Introduction

Let  $f, g$  be two functions defined and integrable on  $[a, b]$ . Assume that

$$\varphi \leq f(x) \leq \Phi \quad \text{and} \quad \gamma \leq g(x) \leq \Gamma$$

for each  $x \in [a, b]$ , where  $\varphi, \Phi, \gamma, \Gamma$  are given real constants. Then we have the following inequality which is well known in the literature as the Grüss inequality ([5, pp. 296])

$$\left| \frac{1}{b-a} \int_a^b f(x) g(x) dx - \frac{1}{b-a} \int_a^b f(x) dx \cdot \frac{1}{b-a} \int_a^b g(x) dx \right| \leq \frac{1}{4} |\Phi - \varphi| \cdot |\Gamma - \gamma|.$$

In this inequality, G. Grüss has proven that the constant  $\frac{1}{4}$  is the best possible in the sense that it cannot be replaced by a smaller one, and this is achieved when

$$f(x) = g(x) = \operatorname{sgn} \left( x - \frac{a+b}{2} \right).$$

Recently, S.S. Dragomir proved the following Grüss' type inequality in real or complex inner product spaces [1].

**Theorem 294** *Let  $(H, \langle \cdot, \cdot \rangle)$  be an inner product space over  $\mathbb{K}$  ( $\mathbb{K} = \mathbb{R}, \mathbb{C}$ ) and  $e \in H$ ,  $\|e\| = 1$ . If  $\varphi, \gamma, \Phi, \Gamma$  are real or complex numbers and  $x, y$  are vectors in  $H$  such that the conditions*

$$\operatorname{Re} \langle \Phi e - x, x - \varphi e \rangle \geq 0 \quad \text{and} \quad \operatorname{Re} \langle \Gamma e - x, x - \gamma e \rangle \geq 0$$

*hold, then we have the inequality*

$$|\langle x, y \rangle - \langle x, e \rangle \langle e, y \rangle| \leq \frac{1}{4} |\Phi - \varphi| \cdot |\Gamma - \gamma|. \quad (6.37)$$

*The constant  $\frac{1}{4}$  is best possible in sense that it cannot be replaced by a smaller constant.*

In [2], by using the following lemmas

**Lemma 295** *Let  $x, e \in H$  with  $\|e\| = 1$  and  $\delta, \Delta \in \mathbb{K}$  with  $\delta \neq \Delta$ . Then*

$$\operatorname{Re} \langle \Delta e - x, x - \delta e \rangle \geq 0$$

*if and only if*

$$\left\| x - \frac{\delta + \Delta}{2} e \right\| \leq \frac{1}{2} |\Delta - \delta|.$$

and

**Lemma 296** *Let  $x, e \in H$  with  $\|e\| = 1$ . Then one has the following representation*

$$0 \leq \|x\|^2 - |\langle x, e \rangle|^2 = \inf_{\lambda \in \mathbb{K}} \|x - \lambda e\|^2.$$

the author gave an alternative proof for (6.37) and also obtained the following refinement of it, namely

**Theorem 297** *Let  $(H, \langle \cdot, \cdot \rangle)$  be an inner product space over  $\mathbb{K}$  ( $\mathbb{K} = \mathbb{R}, \mathbb{C}$ ) and  $e \in H, \|e\| = 1$ . If  $\varphi, \gamma, \Phi, \Gamma$  are real or complex numbers and  $x, y$  are vectors in  $H$  such that either the conditions*

$$\operatorname{Re} \langle \Phi e - x, x - \varphi e \rangle \geq 0, \quad \operatorname{Re} \langle \Gamma e - y, y - \gamma e \rangle \geq 0,$$

*or equivalently,*

$$\left\| x - \frac{\varphi + \Phi}{2} \cdot e \right\| \leq \frac{1}{2} |\Phi - \varphi|, \quad \left\| y - \frac{\gamma + \Gamma}{2} \cdot e \right\| \leq \frac{1}{2} |\Gamma - \gamma|,$$

*hold, then we have the inequality*

$$\begin{aligned} & |\langle x, y \rangle - \langle x, e \rangle \langle e, y \rangle| \\ & \leq \frac{1}{4} |\Phi - \varphi| \cdot |\Gamma - \gamma| - [\operatorname{Re} \langle \Phi e - x, x - \varphi e \rangle]^{\frac{1}{2}} [\operatorname{Re} \langle \Gamma e - y, y - \gamma e \rangle]^{\frac{1}{2}} \\ & \leq \left( \frac{1}{4} |\Phi - \varphi| \cdot |\Gamma - \gamma| \right). \end{aligned}$$

*The constant  $\frac{1}{4}$  is best possible.*

Further, as a generalization for orthonormal families of vectors in inner product spaces, S.S. Dragomir proved, in [3], the following reverse of Bessel's inequality:

**Theorem 298** *Let  $\{e_i\}, i \in I$  be a family of orthonormal vectors in  $H$ ,  $F$  a finite part of  $I$ ,  $\varphi_i, \Phi_i \in \mathbb{K}, i \in F$  and  $x$  a vector in  $H$  such that either the condition*

$$\operatorname{Re} \left\langle \sum_{i \in F} \Phi_i e_i - x, x - \sum_{i \in F} \varphi_i e_i \right\rangle \geq 0,$$

*or equivalently,*

$$\left\| x - \sum_{i \in F} \frac{\Phi_i + \varphi_i}{2} e_i \right\| \leq \frac{1}{2} \left( \sum_{i \in F} |\Phi_i - \varphi_i|^2 \right)^{\frac{1}{2}},$$

holds, then we have the following reverse of Bessel's inequality

$$\begin{aligned} \|x\|^2 - \sum_{i \in F} |\langle x, e_i \rangle|^2 \\ \leq \frac{1}{4} \sum_{i \in F} |\Phi_i - \varphi_i|^2 - \sum_{i \in F} \left| \frac{\varphi_i + \Phi_i}{2} - \langle x, e_i \rangle \right|^2. \end{aligned} \quad (6.38)$$

The constant  $\frac{1}{4}$  is best possible.

The corresponding Grüss type inequality is embodied in the following theorem:

**Theorem 299** Let  $\{e_i\}_{i \in I}$  be a family of orthonormal vectors in  $H$ ,  $F$  a finite part of  $I$ ,  $\phi_i, \gamma_i, \Phi_i, \Gamma_i \in \mathbb{R}$  ( $i \in F$ ), and  $x, y \in H$ . If either

$$\begin{aligned} \operatorname{Re} \left\langle \sum_{i=1}^n \Phi_i e_i - x, x - \sum_{i=1}^n \phi_i e_i \right\rangle &\geq 0, \\ \operatorname{Re} \left\langle \sum_{i=1}^n \Gamma_i e_i - y, y - \sum_{i=1}^n \gamma_i e_i \right\rangle &\geq 0, \end{aligned}$$

or equivalently,

$$\begin{aligned} \left\| x - \sum_{i \in F} \frac{\Phi_i + \phi_i}{2} e_i \right\| &\leq \frac{1}{2} \left( \sum_{i \in F} |\Phi_i - \phi_i|^2 \right)^{\frac{1}{2}}, \\ \left\| y - \sum_{i \in F} \frac{\Gamma_i + \gamma_i}{2} e_i \right\| &\leq \frac{1}{2} \left( \sum_{i \in F} |\Gamma_i - \gamma_i|^2 \right)^{\frac{1}{2}}, \end{aligned}$$

hold true, then

$$\begin{aligned} 0 &\leq \left| \langle x, y \rangle - \sum_{i=1}^n \langle x, e_i \rangle \langle e_i, y \rangle \right| \\ &\leq \frac{1}{4} \left( \sum_{i=1}^n |\Phi_i - \phi_i|^2 \right)^{\frac{1}{2}} \cdot \left( \sum_{i=1}^n |\Gamma_i - \gamma_i|^2 \right)^{\frac{1}{2}} \\ &\quad - \sum_{i \in F} \left| \frac{\Phi_i + \phi_i}{2} - \langle x, e_i \rangle \right| \left| \frac{\Gamma_i + \gamma_i}{2} - \langle y, e_i \rangle \right| \\ &\left( \leq \frac{1}{4} \left( \sum_{i=1}^n |\Phi_i - \phi_i|^2 \right)^{\frac{1}{2}} \cdot \left( \sum_{i=1}^n |\Gamma_i - \gamma_i|^2 \right)^{\frac{1}{2}} \right). \end{aligned}$$

The constant  $\frac{1}{4}$  is best possible in the sense that it cannot be replaced by a smaller constant.

The main aim here is to provide some similar inequalities which, giving refinements of the usual Grüss' inequality, are known in the literature as pre-Grüss type inequalities. Applications for Lebesgue integrals in general measure spaces are also given.

### 6.6.2 Pre-Grüss Inequalities in Inner Product Spaces

We start with the following result [4]:

**Theorem 300** *Let  $(H, \langle \cdot, \cdot \rangle)$  be an inner product space over  $\mathbb{K}$ , ( $\mathbb{K} = \mathbb{R}, \mathbb{C}$ ) and  $e \in H$ ,  $\|e\| = 1$ . If  $\varphi, \Phi$  are real or complex numbers and  $x, y$  are vectors in  $H$  such that either the condition*

$$\operatorname{Re} \langle \Phi e - x, x - \varphi e \rangle \geq 0,$$

*or equivalently,*

$$\left\| x - \frac{\varphi + \Phi}{2} e \right\| \leq \frac{1}{2} |\Phi - \varphi|, \quad (6.39)$$

*holds true, then we have the inequalities*

$$|\langle x, y \rangle - \langle x, e \rangle \langle e, y \rangle| \leq \frac{1}{2} |\Phi - \varphi| \cdot \sqrt{(\|y\|^2 - |\langle y, e \rangle|^2)} \quad (6.40)$$

*and*

$$\begin{aligned} & |\langle x, y \rangle - \langle x, e \rangle \langle e, y \rangle| \\ & \leq \frac{1}{2} |\Phi - \varphi| \cdot \|y\| - (\operatorname{Re} \langle \Phi e - x, x - \varphi e \rangle)^{\frac{1}{2}} \cdot |\langle y, e \rangle|. \end{aligned} \quad (6.41)$$

**Proof.** It is obvious that:

$$\langle x, y \rangle - \langle x, e \rangle \langle e, y \rangle = \langle x - \langle x, e \rangle e, y - \langle y, e \rangle e \rangle.$$

Using Schwarz's inequality in inner product spaces  $|\langle u, v \rangle| \leq \|u\| \cdot \|v\|$  for the vectors  $x - \langle x, e \rangle e$  and  $y - \langle y, e \rangle e$ , we deduce:

$$|\langle x, y \rangle - \langle x, e \rangle \langle e, y \rangle|^2 \leq (\|x\|^2 - |\langle x, e \rangle|^2) \cdot (\|y\|^2 - |\langle y, e \rangle|^2). \quad (6.42)$$

Now, the inequality (6.40) is a simple consequence of (6.37) for  $x = y$ , or of Lemma 296 and (6.39).

Since (see for instance [1]),

$$\begin{aligned} & \|x\|^2 - |\langle x, e \rangle|^2 \\ & = \operatorname{Re} ((\Phi - \langle x, e \rangle) \cdot (\langle e, x \rangle - \bar{\varphi})) - \operatorname{Re} \langle \Phi e - x, x - \varphi e \rangle, \end{aligned} \quad (6.43)$$

then making use of the elementary inequality  $4 \operatorname{Re} (a\bar{b}) \leq |a + b|^2$  with  $a, b \in \mathbb{K}$  ( $\mathbb{K} = \mathbb{R}, \mathbb{C}$ ), we can state that

$$\operatorname{Re} ((\Phi - \langle x, e \rangle) \cdot (\langle e, x \rangle - \bar{\varphi})) \leq \frac{1}{4} |\Phi - \varphi|^2. \quad (6.44)$$

Using (6.43) and (6.44) we have

$$\|x\|^2 - |\langle x, e \rangle|^2 \leq \left( \frac{1}{2} |\Phi - \varphi| \right)^2 - \left( (\operatorname{Re} \langle \Phi e - x, x - \varphi e \rangle)^{\frac{1}{2}} \right)^2. \quad (6.45)$$

Taking into account the inequalities (6.42) and (6.45), we get that

$$\begin{aligned} & |\langle x - \langle x, e \rangle e, y - \langle y, e \rangle e \rangle|^2 \\ & \leq \left( \left( \frac{1}{2} |\Phi - \varphi| \right)^2 - \left( (\operatorname{Re} \langle \Phi e - x, x - \varphi e \rangle)^{\frac{1}{2}} \right)^2 \right) \cdot \left( \|y\|^2 - |\langle y, e \rangle|^2 \right). \end{aligned}$$

Finally, using the elementary inequality for positive real numbers:

$$(m^2 - n^2) \cdot (p^2 - q^2) \leq (mp - nq)^2, \quad (6.46)$$

we have:

$$\begin{aligned} & \left( \left( \frac{1}{2} |\Phi - \varphi| \right)^2 - \left( (\operatorname{Re} \langle \Phi e - x, x - \varphi e \rangle)^{\frac{1}{2}} \right)^2 \right) \cdot \left( \|y\|^2 - |\langle y, e \rangle|^2 \right) \\ & \leq \left( \frac{1}{2} |\Phi - \varphi| \cdot \|y\| - (\operatorname{Re} \langle \Phi e - x, x - \varphi e \rangle)^{\frac{1}{2}} \cdot |\langle y, e \rangle| \right)^2, \end{aligned}$$

giving the desired inequality (6.41). ■

A similar version for Bessel's inequality is incorporated in the following theorem [4]:

**Theorem 301** *Let  $\{e_i\}_{i \in I}$ , be a family of orthonormal vectors in  $H$ ,  $F$  a finite part of  $I$ ,  $\varphi_i, \Phi_i \in \mathbb{K}$ ,  $i \in F$  and  $x, y$  vectors in  $H$  such that either the condition*

$$\operatorname{Re} \left\langle \sum_{i \in F} \Phi_i e_i - x, x - \sum_{i \in F} \varphi_i e_i \right\rangle \geq 0,$$

*or equivalently,*

$$\left\| x - \sum_{i \in F} \frac{\Phi_i + \varphi_i}{2} e_i \right\| \leq \frac{1}{2} \left( \sum_{i \in F} |\Phi_i - \varphi_i|^2 \right)^{\frac{1}{2}}$$

*holds. Then we have inequalities*

$$\begin{aligned} & \left| \langle x, y \rangle - \sum_{i \in F} \langle x, e_i \rangle \langle e_i, y \rangle \right| \\ & \leq \frac{1}{2} \left( \sum_{i \in F} |\Phi_i - \varphi_i|^2 \right)^{\frac{1}{2}} \sqrt{\left( \|y\|^2 - \sum_{i \in F} |\langle y, e_i \rangle|^2 \right)} \quad (6.47) \end{aligned}$$



and

$$\begin{aligned} \left| \langle x, y \rangle - \sum_{i \in F} \langle x, e_i \rangle \langle e_i, y \rangle \right| &\leq \frac{1}{2} \left( \sum_{i \in F} |\Phi_i - \varphi_i|^2 \right)^{\frac{1}{2}} \cdot \|y\| \\ &\quad - \left( \sum_{i \in F} \left| \frac{\Phi_i + \varphi_i}{2} - \langle x, e_i \rangle \right|^2 \right)^{\frac{1}{2}} \cdot \left( \sum_{i \in F} |\langle y, e_i \rangle|^2 \right)^{\frac{1}{2}}. \end{aligned} \quad (6.48)$$

**Proof.** It is obvious (see for example [3]) that:

$$\langle x, y \rangle - \sum_{i \in F} \langle x, e_i \rangle \langle e_i, y \rangle = \left\langle x - \sum_{i \in F} \langle x, e_i \rangle e_i, y - \sum_{i \in F} \langle y, e_i \rangle e_i \right\rangle.$$

Using Schwarz's inequality in inner product spaces, we have:

$$\begin{aligned} &\left| \left\langle x - \sum_{i \in F} \langle x, e_i \rangle e_i, y - \sum_{i \in F} \langle y, e_i \rangle e_i \right\rangle \right|^2 \\ &\leq \left\| x - \sum_{i \in F} \langle x, e_i \rangle e_i \right\|^2 \cdot \left\| y - \sum_{i \in F} \langle y, e_i \rangle e_i \right\|^2 \\ &= \left( \|x\|^2 - \sum_{i \in F} |\langle x, e_i \rangle|^2 \right) \cdot \left( \|y\|^2 - \sum_{i \in F} |\langle y, e_i \rangle|^2 \right). \end{aligned} \quad (6.49)$$

In a similar manner to the one in the proof of Theorem 300 we may conclude that (6.47) holds true.

Now, using (6.38) and (6.49) we also have:

$$\begin{aligned} &\left| \left\langle x - \sum_{i \in F} \langle x, e_i \rangle e_i, y - \sum_{i \in F} \langle y, e_i \rangle e_i \right\rangle \right|^2 \\ &\leq \left( \frac{1}{2} \left( \left( \sum_{i \in F} |\Phi_i - \varphi_i|^2 \right)^{\frac{1}{2}} \right)^2 - \left( \left( \sum_{i \in F} \left| \frac{\varphi_i + \Phi_i}{2} - \langle x, e_i \rangle \right|^2 \right)^{\frac{1}{2}} \right)^2 \right) \\ &\quad \times \left( \|y\|^2 - \sum_{i \in F} |\langle y, e_i \rangle|^2 \right). \end{aligned}$$

Finally, utilizing the elementary inequality (6.46), we have

$$\begin{aligned} & \left( \frac{1}{2} \left( \left( \sum_{i \in F} |\Phi_i - \varphi_i|^2 \right)^{\frac{1}{2}} \right)^2 - \left( \left( \sum_{i \in F} \frac{\varphi_i + \Phi_i}{2} - \langle x, e_i \rangle \right)^2 \right)^{\frac{1}{2}} \right)^2 \\ & \times \left( \|y\|^2 - \sum_{i \in F} |\langle y, e_i \rangle|^2 \right) \leq \left( \frac{1}{2} \left( \sum_{i \in F} |\Phi_i - \varphi_i|^2 \right)^{\frac{1}{2}} \cdot \|y\|^2 \right. \\ & \left. - \left( \sum_{i \in F} \left| \frac{\varphi_i + \Phi_i}{2} - \langle x, e_i \rangle \right|^2 \right)^{\frac{1}{2}} \cdot \sum_{i \in F} |\langle y, e_i \rangle|^2 \right)^2, \quad (6.50) \end{aligned}$$

which gives the desired result (6.48). ■

### 6.6.3 Applications for Integrals

Let  $(\Omega, \Sigma, \mu)$  be a measure space consisting of a set  $\Omega$ ,  $\Sigma$  a  $\sigma$ -algebra of parts and  $\mu$  a countably additive and positive measure on  $\Sigma$  with values in  $\mathbb{R} \cup \{\infty\}$ . Denote by  $L^2(\Omega, \mathbb{K})$  the Hilbert space of all real or complex valued functions  $f$  defined on  $\Omega$  and 2-integrable on  $\Omega$ , i. e.

$$\int_{\Omega} |f(s)|^2 d\mu(s) < \infty.$$

The following proposition holds [4].

**Proposition 302** *If  $f, g, h \in L^2(\Omega, \mathbb{K})$  and  $\varphi, \Phi \in \mathbb{K}$  are such that  $\int_{\Omega} |h(s)|^2 d\mu(s) = 1$  and, either*

$$\int_{\Omega} \operatorname{Re}((\Phi h(s) - f(s))(\bar{f}(s) - \bar{\varphi} \bar{h}(s))) d\mu(s) \geq 0, \quad (6.51)$$

*or equivalently,*

$$\left( \int_{\Omega} \left| f(s) - \frac{\Phi + \varphi}{2} h(s) \right|^2 d\mu(s) \right)^{\frac{1}{2}} \leq \frac{1}{2} |\Phi - \varphi|$$

*holds, then we have the inequalities*

$$\begin{aligned} & \left| \int_{\Omega} f(s) \bar{g}(s) d\mu(s) - \int_{\Omega} f(s) \bar{h}(s) d\mu(s) \int_{\Omega} h(s) \bar{g}(s) d\mu(s) \right| \\ & \leq \frac{1}{2} |\Phi - \varphi| \cdot \sqrt{\left( \int_{\Omega} |g(s)|^2 d\mu(s) - \left| \int_{\Omega} h(s) \bar{g}(s) d\mu(s) \right|^2 \right)} \end{aligned}$$

and

$$\begin{aligned} & \left| \int_{\Omega} f(s) \bar{g}(s) d\mu(s) - \int_{\Omega} f(s) \bar{h}(s) d\mu(s) \int_{\Omega} h(s) \bar{g}(s) d\mu(s) \right| \\ & \leq \frac{1}{2} |\Phi - \varphi| \cdot \left( \int_{\Omega} |g(s)|^2 d\mu(s) \right)^{\frac{1}{2}} \\ & - \left( \int_{\Omega} \operatorname{Re}((\Phi h(s) - f(s)) (h(s) \bar{f}(s) - \varphi h(s))) d\mu(s) \right)^{\frac{1}{2}} \left| \int_{\Omega} h(s) \bar{g}(s) d\mu(s) \right|. \end{aligned}$$

**Proof.** The proof follows by Theorem 300 on choosing  $H = L^2(\Omega, \mathbb{K})$  with the inner product

$$\langle f, g \rangle = \int_{\Omega} f(s) \bar{g}(s) d\mu(s).$$

■

**Remark 303** We observe that a sufficient condition for (6.51) to hold is:

$$\operatorname{Re}(\Phi h(s) - f(s)) (\bar{f}(s) - \overline{\varphi h}(s)) \geq 0, \quad (6.52)$$

for  $\mu$ -a.e.  $s \in \Omega$ .

If the functions are real-valued, then, for  $\Phi$  and  $\varphi$  real numbers, a sufficient condition for (6.52) to hold is

$$\Phi h(s) \geq f(s) \geq \varphi h(s)$$

for  $\mu$ -a.e.  $s \in \Omega$ .

In this way we can see the close connection that exists between the classical Grüss inequality and the results obtained above.

Now, consider the family  $\{f_i\}_{i \in I}$  of functions in  $L^2(\Omega, \mathbb{K})$  with the properties that

$$\int_{\Omega} f_i(s) \overline{f_j(s)} d\mu(s) = \delta_{ij}, \quad i, j \in I,$$

where  $\delta_{ij}$  is 0 if  $i \neq j$  and  $\delta_{ij} = 1$  if  $i = j$ .  $\{f_i\}_{i \in I}$  is an orthonormal family in  $L^2(\Omega, \mathbb{K})$ .

The following proposition holds [4].

**Proposition 304** Let  $\{f_i\}_{i \in I}$  be an orthonormal family of functions in  $L^2(\Omega, \mathbb{K})$ ,  $F$  a finite subset of  $I$ ,  $\phi_i, \Phi_i \in \mathbb{K}$  ( $i \in F$ ) and  $f \in L^2(\Omega, \mathbb{K})$ , such that either

$$\int_{\Omega} \operatorname{Re} \left[ \left( \sum_{i \in F} \Phi_i f_i(s) - f(s) \right) \left( \overline{f}(s) - \sum_{i \in F} \overline{\phi_i} \overline{f_i(s)} \right) \right] d\mu(s) \geq 0 \quad (6.53)$$

or equivalently,

$$\int_{\Omega} \left| f(s) - \sum_{i \in F} \frac{\Phi_i + \phi_i}{2} f_i(s) \right|^2 d\mu(s) \leq \frac{1}{4} \sum_{i \in F} |\Phi_i - \phi_i|^2.$$

holds. Then we have the inequalities

$$\begin{aligned} & \left| \int_{\Omega} f(s) \overline{g(s)} d\mu(s) - \sum_{i \in F} \int_{\Omega} f(s) \overline{f_i(s)} d\mu(s) \int_{\Omega} f_i(s) \overline{g(s)} d\mu(s) \right| \\ & \leq \frac{1}{2} \left( \sum_{i \in F} |\Phi_i - \phi_i|^2 \right)^{\frac{1}{2}} \left( \int_{\Omega} |g(s)|^2 d\mu(s) - \sum_{i \in F} \left| \int_{\Omega} g(s) \overline{f_i(s)} d\mu(s) \right|^2 \right)^{\frac{1}{2}} \end{aligned}$$

and

$$\begin{aligned} & \left| \int_{\Omega} f(s) \overline{g(s)} d\mu(s) - \sum_{i \in F} \int_{\Omega} f(s) \overline{f_i(s)} d\mu(s) \int_{\Omega} f_i(s) \overline{g(s)} d\mu(s) \right| \\ & \leq \frac{1}{2} \left( \sum_{i \in F} |\Phi_i - \phi_i|^2 \right)^{\frac{1}{2}} \left( \int_{\Omega} |g(s)|^2 d\mu(s) \right)^{\frac{1}{2}} \\ & - \left( \sum_{i \in F} \left| \frac{\Phi_i + \phi_i}{2} - \int_{\Omega} f(s) \overline{f_i(s)} d\mu(s) \right|^2 \right)^{\frac{1}{2}} \left( \sum_{i \in F} \left| \int_{\Omega} f(s) \overline{f_i(s)} d\mu(s) \right|^2 \right)^{\frac{1}{2}}. \end{aligned}$$

The proof is obvious by Theorem 300 and we omit the details.

**Remark 305** In the real case, we observe that a sufficient condition for (6.53) to hold is

$$\sum_{i \in F} \Phi_i f_i(s) \geq f(s) \geq \sum_{i \in F} \varphi_i f_i(s)$$

for  $\mu$ -a.e.  $s \in \Omega$ .

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